EQUAL RANK EMBEDDING AND ITS RELATED CONSTRUCTION TO SUPERCONFORMAL FIELD THEORIES

By

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To my father, my mother, my brothers,
my sisters, and my love.

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EQUAL RANK EMBEDDING AND ITS RELATED CONSTRUCTION TO SUPERCONFORMAL FIELD THEORIES

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We have studied the GKRS construction of an Euler multiplet of semisimple complex Lie algebras and extended the construction to basic Lie superalgebras. We have observed that the Kostant's cubic Dirac operators of these two algebras are in a similar form and, in the case of Lie algebras, are analog to the supercurrents in superconformal field theories. At the level of representation theory, we have studied the analogy between the GKRS construction of the Euler multiplet of semisimple complex Lie algebras and the Kazama-Suzuki construction of super-Kac-Moody algebras associated with superconformal field theories. The Euler multiplets constructed by the GKRS method were found to be the intrinsic ground states of superconformal theories. The minimum eigenvalues of the Kostant operator in the GKRS construction and of energy-momentum tensor in the Kazama-Suzuki construction were derived from the Euler multiplet and were shown to be zero. We also studied the gauged supersymmetric Wess-Zumino-Witten Lagrangian and derived the superconformal generators, supercurrents and energy-momentum tensors, in the $SU(2)/U(1)$ case.

CHAPTER 1
INTRODUCTION

The study of some patterns [1], connected with $N = 1$ supergravity theory in eleven dimensions, has recently been understood in terms of a Weyl character formula by GKRS [2], based on equal rank embeddings of reductive subalgebras[1] of semisimple complex Lie algebras. By using the GKRS construction, all possible equal rank embeddings were cataloged, and nearly all supersymmetric multiplets of massless and massive particles which have already known in supersymmetric gauge field theories emerge as the lowest lines of the infinite tower Euler multiplet spectra, some of them shown in details in Ref. [1]. Immediately after the appearance of the Weyl character formula for equal rank embedding as an index formula for the Dirac operator, Kostant moved the subject forward and related the Euler multiplet spectra to the kernel of a cubic Dirac operator [3].

In this thesis, we extend the GKRS character formula and Kostant's cubic Dirac operator for equal rank embedding of semisimple complex Lie algebras to those of basic Lie superalgebras [4]. In chapter 2, we give a brief review of the GKRS construction and of Kostant's cubic Dirac operator. The minimum eigenvalue of the square of the Kostant operator is derived from an Euler multiplet. The lowest lines of the Euler multiplets are classified by $SO(\text{dim } g/h)$. $SO(16)$ triplets as an exception to the GKRS construction are shown. In chapter 3, we apply the GKRS construction to basic Lie superalgebras. The Kac character formulas are derived in terms of equal rank subalgebras. We give a simple and explicit formulation for a typical representation of type I Lie superalgebras and build a multiplet of type I Lie superalgebras from that of semisimple complex Lie algebras. Kostant's cubic Dirac operator is constructed for full Lie superalgebras and then for equal rank embeddings. In chapter 4, the superconformal field theories according to the Kazama-Suzuki construction [5] are classified by the equal rank embeddings. We show that at the level of representation theory there is a correspondence between the GKRS construction and the Kazama-Suzuki construction. The Euler multiplets constructed by the GKRS method are shown to be the intrinsic ground states of the superconformal theories. The minimum eigenvalue of energy-momentum tensor in the Kazama-Suzuki construction is derived from the Euler multiplet and is shown to be zero. In chapter 5, we study the gauged supersymmetric Wess-Zumino-Witten action of $SU(2)/U(1)$ model and derive the superconformal generators from the gauge-fixed Lagrangian. Up to some multiplicative factors, the superconformal generators turn out to be similar to those constructed by Kazama and Suzuki.

---

[1] A reductive Lie algebra is the direct sum of semisimple Lie algebras and abelian Lie algebras.

CHAPTER 2
GKRS CONSTRUCTION OF AN EULER MULTIPLET: SEMISIMPLE COMPLEX LIE ALGEBRAS

In this chapter, we review the GKRS construction of an Euler multiplet of a semisimple complex Lie algebra in section 2.1 and Kostant's cubic Dirac operator in section 2.2. The minimum eigenvalue of the square of the Kostant operatoris derived from an Euler multiplet in section 2.3. The lowest lines of the Euler multiplets are classified according to $SO(\dim g/h)$ in section 2.4 and $SO(16)$ triplets that are exception to the GKRS construction are briefly mentioned in section 2.5.

2.1 GKRS character formula of semisimple Lie algebras

Let h be a complex reductive subalgebra of a semisimple complex Lie algebra g. Assume that both g and h have the same rank. Let $\mathbf{C}$ be order of C, the quotient of the Weyl group of g by that of h. The restricted conditions for this kind of equal rank embedding, $g \supset h$, are that (1) positive roots of g must contain those of h, i.e. $\Phi^+(g) \supset \Phi^+(h)$, and (2) the simple roots of g and h must be chosen consistently so that the positive Weyl chamber of h contains that of g. In the Cartan-Weyl basis, the elements c of C, the quotient of the Weyl group $W(g)/W(h)$, acting on the sum of highest weight Λ and ρ_g, the Weyl vector of g, and then subtracting by ρ_h, the Weyl vector of h, generate $\mathbf{C}$ irreducible representations of h, called a

C-multiplet or an Euler multiplet,

$$\{c \bullet \Lambda := c(\Lambda + \rho_g) - \rho_h\}.$$

The Weyl character formula of the irreducible representation of g, V_Λ, can be rewritten in terms of the irreducible representation of h, $U_{c\bullet\Lambda}$, as follows:

$$\begin{aligned} \chi(V_\Lambda) &= \frac{\sum_{w\in W(g)} \mathrm{sgn}(w) e^{w(\Lambda+\rho_g)}}{\sum_{w\in W(g)} \mathrm{sgn}(w) e^{w\rho_g}} \\ &= \frac{\sum_{c\in C} \mathrm{sgn}(c) \left(\sum_{w_h\in W(h)} \mathrm{sgn}(w_h) e^{w_h(c\bullet\Lambda+\rho_h)}\right)}{\left(\prod_{\phi\in\Phi^+(g/h)}(e^{\frac{\phi}{2}} - e^{-\frac{\phi}{2}})\right)\left(\prod_{\phi\in\Phi^+(h)}(e^{\frac{\phi}{2}} - e^{-\frac{\phi}{2}})\right)} \\ &= \frac{1}{\Delta}\sum_{c\in C} \mathrm{sgn}(c)\chi(U_{c\bullet\Lambda}), \end{aligned} \tag{2.1}$$

where $\Phi^+(g/h)$ is the set of roots in $\Phi^+(g)$ not contained in $\Phi^+(h)$ and Δ is the character difference of two spinor representations, S^+ and S^-, of $SO(\dim g/h)$, i.e.

$$\Delta := \prod_{\phi\in\Phi^+(g/h)} (e^{\frac{\phi}{2}} - e^{-\frac{\phi}{2}}) = \chi(S^+) - \chi(S^-).$$

The beauty of (2.1) is that it gives us

$$V_\Lambda \otimes S^+ - V_\Lambda \otimes S^- = \sum_{c\in C} \mathrm{sgn}(c) U_{c\bullet\Lambda}. \tag{2.2}$$

(2.2) should be viewed as an equation in the Grothendieck ring of h. The lhs is the character of the difference between two representations of the same dimension: the representation of g with highest weight Λ restricted to h and then tensored with each of the half spin representations of $SO(\dim g/h)$ restricted to h. In the other words, the lhs is the algebraic index of the Dirac operator associated to Λ and the two half spin representations. The alternating sum on the rhs is just the dimension difference between kernel and cokernel of the Dirac operator. All the representations on the rhs are inequivalent and so the rhs is the end result of a lot of cancellation in the lhs, in short an index formula for the Dirac operator. The remarkable consequences of (2.2) are that a number of irreducible representations left from the difference of $V_\Lambda \otimes S^+$ and $V_\Lambda \otimes S^-$ is always equal to $\mathbf{C}$, independent of Λ, and the multiplicity of each $U_{c\bullet\Lambda}$ representation is exactly one.

Notice that $\mathbf{C}$ is an Euler number, a topological invariant of a coset manifold G/H where G is the simply connected and compact group whose complexified Lie algebra is g and H is the subgroup of G associated with h. Now, we would like to give some examples of the coset manifolds where supersymmetric multiplets appear to be in their lowest lines of the infinite tower multiplet spectra. $N = 2$ hypermultiplet, $N = 4$ vector multiplet, and $N = 8$ supergravity multiplet which undoubtedly emerge in the lowest lines of $SU(N+1) \supset SU(N) \times U(1)$ and $SO(N+2) \supset SO(N) \times SO(2)$ series live on the N-dimensional complex projective space and the Grassmannian manifold of 2-planes in R^{N+2} (which is also an N-dimensional complex manifold), respectively. If $SO(2)$ or $U(1)$ is viewed as the light-cone little group, these lowest line spectra are massless supermultiplets in 4-dimensional space-time. Whereas $N = 1$, $N = 2$, $N = 3$, and $N = 4$ massive (massless) multiplets in 4-dimensional (6-dimensional) space-time which emerge in the lowest lines of $Sp(2N + 2) \supset Sp(2N) \times Sp(2)$ series live on N-dimensional quaternionic projective space. The last multiplet that we would like to mention is the $N = 1$ massless (massive) supergravity triplet in 11-dimensional (10-dimensional) space-time. The triplet emerges from $F_4 \supset SO(9)$ and lives on the (16-dimensional) Cayley plane. All infinite tower multiplet spectra are a kernel of Kostant's cubic Dirac operator $\not{\mathcal{K}}$ [3, 6]

$$\mathrm{Ker}(\not{\mathcal{K}}_\Lambda^2) = \mathrm{Ker}(\not{\mathcal{K}}_\Lambda) = \sum_{c \in C} U_{c \bullet \Lambda}.$$

Note that the kernel of Kostant's Dirac operator is the unsigned sum of the $U_{c\bullet\Lambda}$'s. To recover the signs on the rhs of (2.2), one must take the index (i.e., the difference between kernel and cokernel) of the Kostant's cubic operator, viewed as a map from the positive to negative half-spin spaces.

2.2 Kostant operator of semisimple Lie algebras

Let J's be generators of a semisimple complex Lie algebra g such that

$$[J_A, J_B] = i f_{[ABC]} J_C, \tag{2.3}$$

where indices $A, B, C = 1 \ldots \dim g$. A basis of the generators is chosen so that these generators are orthonormal with respect to an inner product on g that is invariant under the adjoint action. In the case that g is semisimple, the inner product on g is the Killing form. Let Λ be in the positive Weyl chamber of g and let J_A^Λ be the image of J_A under the irreducible representation of g with highest weight Λ. Kostant's cubic Dirac operator of the pair (g, Λ) is the operator on

$V_\Lambda \otimes S$ [1] given by

$$\not{K}_g(\Lambda) = \gamma_A J^\Lambda_A - \frac{i}{2} f_{[ABC]} \gamma_{[A} \gamma_B \gamma_{C]}, \tag{2.4}$$

where the Dirac $\gamma-$matrices satisfy the Clifford algebra

$$\{\gamma_A, \gamma_B\} = 2\delta_{AB}. \tag{2.5}$$

Squaring (2.4) gives us

$$\not{K}^2_g(\Lambda) = (J^\Lambda_A)^2 + (\frac{i}{2} f_{[ABC]} \gamma_{[A} \gamma_B \gamma_{C]})^2 = C_2(\Lambda) + |\rho_g|^2, \tag{2.6}$$

where $C_2(\Lambda) = (\Lambda, \Lambda + \rho_g)$ is the quadratic Casimir eigenvalue of the irreducible representation Λ and ρ_g is one-half the sum of positive roots of g. Both $C_2(\Lambda)$ and $|\rho_g|^2$ are calculated in the orthornormal basis $\{e_i\}$ where $(e_i, e_j) = \delta_{ij}$. The eigenvalue of (2.6) is always positive definite. Therefore, $\not{K}^2_g$ is a positive eigenvalue operator.

In the case of the quotient g/h, the generators $J_i, i = 1 \ldots \dim h$, lie in g and the generators $J_a, a = h + 1 \ldots \dim g$, are a basis of the orthogonal complement of h with respect to the Killing form on g. Both J_i and J_a satisfy the following commutation relations,

$$[J_i, J_j] = i f_{[ijk]} J_k, \tag{2.7}$$

$$[J_i, J_a] = i f_{[iab]} J_b, \tag{2.8}$$

$$[J_a, J_b] = i f_{[abj]} J_j + i f_{[abc]} J_c. \tag{2.9}$$

---

[1] The spinor module S is the direct sum of S^+ and S^-.

The twisted Dirac operator [2] on $h \subset g$ acting on $V_\Lambda \otimes S$ is

$$\not{\mathcal{K}}_h(\Lambda) = \gamma_i\left(J_i^\Lambda - \frac{i}{2} f_{[iab]}\gamma_{[a}\gamma_{b]}\right) - \frac{i}{2} f_{[ijk]}\gamma_{[i}\gamma_j\gamma_{k]}, \tag{2.10}$$

and the Kostant operator of the quotient g/h is

$$\not{\mathcal{K}}_{g/h}(\Lambda) = \not{\mathcal{K}}_g(\Lambda) - \not{\mathcal{K}}_h(\Lambda) = \gamma_a J_a^\Lambda - \frac{i}{2} f_{[abc]}\gamma_{[a}\gamma_b\gamma_{c]}. \tag{2.11}$$

Decompose $V_\Lambda \otimes S$ into irreducible representations of h. Let V_μ be one of these h-representations, where μ is its highest weight. Restricting (2.11) to V_μ and squaring gives a scalar operator:

$$\begin{aligned} (\not{\mathcal{K}}_{g/h}(\Lambda)|_{V_\mu})^2 &= (J_A^\Lambda)^2 - (J_i^\Lambda + S_i)^2 + S_i^2 \\ &= C_2(\Lambda) - C_2(\mu) + |\rho_g|^2 - |\rho_h|^2, \end{aligned} \tag{2.12}$$

where $S_i = -\frac{i}{2} f_{[iab]}\gamma_{[a}\gamma_{b]}$. In the last line of (2.12), $C_2(\Lambda)$ is the quadratic Casimir eigenvalue for g and $C_2(\mu)$ for h. The rhs of equation (2.12) gives the eigenvalues of $(\not{\mathcal{K}}_{g/h}(\Lambda)|_{V_\mu})^2$ acting on $V_\Lambda \otimes S$. Since the Kostant operator is self-adjoint, all of these eigenvalues are non-negative. We will show in the next section that when $\mu = c \bullet \Lambda$, the eigenvalue (2.12) is minimum and equal to zero:

$$C_2(\Lambda) - C_2(\mu) + |\rho_g|^2 - |\rho_h|^2 = \begin{cases} 0 \text{ if } \mu = c \bullet \Lambda \\ \text{positive otherwise.} \end{cases}$$

---

[2] This operator first appeared in the Kazama and Suzuki's paper [5].

The set $\{\mu = c \bullet \Lambda\}$ is the kernel of the Kostant operator and is called the Euler multiplet.

The simplest example is the Kostant operator on $SU(2)/U(1)$,

$$\begin{aligned}
\not{\mathcal{K}}_g(\Lambda) &= \sum_{A=1}^{3} \sigma_A J_A^{\Lambda} - \frac{i}{2}\epsilon_{[123]}\sigma_{[1}\sigma_2\sigma_{3]}, & (2.13)\\
\not{\mathcal{K}}_h(\Lambda) &= \sigma_3(J_3^{\Lambda} - \frac{i}{2}\epsilon_{[123]}\sigma_{[1}\sigma_{2]}), & (2.14)\\
\not{\mathcal{K}}_{g/h}(\Lambda) &= \sigma_1 J_1^{\Lambda} + \sigma_2 J_2^{\Lambda}, & (2.15)
\end{aligned}$$

where σ's are the Pauli matrices. Let Λ be the spin-j representation of $SU(2)$. Acting on the subspace of $V_\Lambda \otimes S$ on which J_3 has eigenvalue m and σ_3 has eigenvalue $\pm 1/2$, one can show that

$$(\not{\mathcal{K}}_{g/h}(\Lambda)|_{V_\mu})^2 = \sum_{A=1}^{3}(J_A^{\Lambda})^2 - (J_3^{\Lambda} + S_3)^2 + S_3^2 = j(j+1) - \tilde{m}^2 + \frac{1}{4}, \qquad (2.16)$$

where $S_3 = -\frac{i}{2}\epsilon_{[123]}\sigma_{[1}\sigma_{2]}$ and where $\tilde{m} = m \pm \frac{1}{2}$. In the case that the eigenvalue of $\not{\mathcal{K}}_{g/h}^2$ is equal to zero, $\tilde{m} = \pm(j+1/2)$. When $j = 0$, the trivial one-dimensional representation of $SU(2)$, one yields $\tilde{m} = \pm 1/2$, the lowest line of the Euler multiplets of $SU(2)/U(1)$.

2.3 Eigenvalue of the square of the Kostant operator on g/h

Now, one can look at an eigenvalue of the square of the Kostant operator from the representation theory of Lie algebras. The eigenvalue of (2.12) can be naively derived from the following inequality,

$$\Lambda + \rho_g \geq w^{-1}(\mu + \rho_h), \tag{2.17}$$

where μ is a weight of h appearing in the decomposition of $V_\Lambda \otimes S$ into irreducible representations of h and w is an element in the Weyl group of g such that $w^{-1}(\mu + \rho_h)$ is a dominant weight. Since the Weyl group preserves scalar product, squaring (2.17) yields

$$|\Lambda + \rho_g|^2 \geq |\mu + \rho_h|^2. \tag{2.18}$$

(2.18) gives rise to

$$(\Lambda, \Lambda + 2\rho_g) + |\rho_g|^2 \geq (\mu, \mu + 2\rho_h) + |\rho_h|^2, \tag{2.19}$$

i.e.

$$C_2(\Lambda) - C_2(\mu) + |\rho_g|^2 - |\rho_h|^2 \geq 0. \tag{2.20}$$

When $\mu = c \bullet \Lambda$, the square of Kostant's cubic Dirac operator on g/h, the equation (2.12), has a zero eigenvalue. Since we have equality in (2.12) if and only if $\mu = c \bullet \Lambda$, when $\mu \neq c \bullet \Lambda$, the eigenvalue of $\not{K}^2_{g/h}$ is always positive.

The $F_4/SO(9)$ is given as an example to verify the inequality (2.20). The quadratic Casimir value of the F_4 irreducible representation, $\Lambda = (a_1\ a_2\ a_3\ a_4)$ in an ω-basis [7, 8], is

$$C_2(\Lambda) = b_1(b_1 + 11) + b_2(b_2 + 5) + b_3(b_3 + 3) + b_4(b_4 + 1) \tag{2.21}$$

where

$$\begin{aligned}
b_1 &= a_4 + 3a_3/2 + 2a_2 + a_1, \\
b_2 &= a_3/2 + a_2 + a_1, \\
b_3 &= a_3/2 + a_2, \\
b_4 &= a_3/2,
\end{aligned}$$

and that of $SO(9)$ irreducible representation, $\lambda = (\tilde{a}_1\ \tilde{a}_2\ \tilde{a}_3\ \tilde{a}_4)$ in the ω-basis, is

$$C_2(\lambda) = \tilde{b}_1(\tilde{b}_1 + 7) + \tilde{b}_2(\tilde{b}_2 + 5) + \tilde{b}_3(\tilde{b}_3 + 3) + \tilde{b}_4(\tilde{b}_4 + 1) \tag{2.22}$$

where

$$\begin{aligned}
\tilde{b}_1 &= \tilde{a}_4/2 + \tilde{a}_3 + \tilde{a}_2 + \tilde{a}_1, \\
\tilde{b}_2 &= \tilde{a}_4/2 + \tilde{a}_3 + \tilde{a}_2, \\
\tilde{b}_3 &= \tilde{a}_4/2 + \tilde{a}_3, \\
\tilde{b}_4 &= \tilde{a}_4/2.
\end{aligned}$$

The dimensional formulas of the F_4 and $SO(9)$ irreducible representations are given in the appendix A. From the Freudenthal-de Vries strange formula [7, 8],

$$|\rho_{F_4}|^2 - |\rho_{SO(9)}|^2 = \frac{1}{12}(g^\vee \text{dim } g - h^\vee \text{dim } h) = 18 \tag{2.23}$$

where $g^\vee$ and $h^\vee$ are the dual Coxeter numbers of g and h, respectively. Here, we take the length of the highest root equal to 2 and the dual Coxeter number is the value of the quadratic Casimir operator in the adjoint representation divided by the length of the highest root.

For convenience, let us define m^2 as the eigenvalue of $\not{\mathcal{K}}^2_{F_4/SO(9)}(\Lambda)$ restricted to V_μ

$$m^2 = C_2(\Lambda) - C_2(\mu) + 18. \tag{2.24}$$

For the first level, $\Lambda = (0\ 0\ 0\ 0)$, the trivial one-dimensional irreducible representation of F_4, we obtain

$$(V_\Lambda \otimes S)|_{SO(9)} = \{(2\ 0\ 0\ 0),\ (0\ 0\ 1\ 0),\ (1\ 0\ 0\ 1)\}, \tag{2.25}$$

and $m^2 = 0$ for all three irreps of $SO(9)$. Therefore, the triplet

$$(2\ 0\ 0\ 0) \oplus (0\ 0\ 1\ 0) \oplus (1\ 0\ 0\ 1)$$

is the kernel of Kostant's cubic Dirac operator.

For the second level, $\Lambda = (0\ 0\ 0\ 1)$, the 26-dimensional irrep of F_4, we obtain $(V_\Lambda \otimes S)|_{SO(9)}$ and m^2 as shown in Table 2.1. For this level, the kernel of the Kostant operator is the triplet

$$(3\ 0\ 0\ 0) \oplus (0\ 0\ 1\ 1) \oplus (1\ 0\ 0\ 2).$$

μ	m^2	μ	m^2
(1 0 0 0)	22	(3 0 0 0)	0
(0 0 0 1)	21	(1 1 0 0)	6
(0 1 0 0)	16	(0 1 0 1)	5
(2 0 0 0)	12	(2 0 0 1)	1
(0 0 1 0)	12	(1 0 1 0)	2
(0 0 0 2)	10	(0 0 1 1)	0
(1 0 0 1)	12	(1 0 0 2)	0

Table 2.1.[3] $\mu \in (26 \otimes S)|_{SO(9)}$ and m^2.

For the next level, $\Lambda = (1\ 0\ 0\ 0)$, the 52-dimensional irrep of F_4, we obtain $(V_\Lambda \otimes S)|_{SO(9)}$ and m^2 as shown in Table 2.2. For this level, the kernel of the Kostant operator is the triplet

$$(2\ 1\ 0\ 0) \oplus (0\ 1\ 1\ 0) \oplus (1\ 1\ 0\ 1).$$

---

[3] The multiplicities of $\mu \in (V_\Lambda \otimes S)|_{SO(9)}$ are not shown.

μ	m^2	μ	m^2
(1 0 0 0)	28	(0 1 0 1)	11
(0 0 0 1)	27	(2 0 0 1)	7
(0 1 0 0)	12	(1 0 1 0)	8
(2 0 0 0)	18	(0 0 1 1)	6
(0 0 1 0)	18	(2 1 0 0)	0
(0 0 0 2)	16	(1 0 0 2)	6
(1 0 0 1)	18	(0 1 1 0)	0
(1 1 0 0)	12	(1 1 0 1)	0

Table 2.2. $\mu \in (52 \otimes S)|_{SO(9)}$ and m^2.

The quadratic Casimir eigenvalues of the F_4 irreducible representations and the $SO(9)$ triplets that satisfy the minimum of (2.20) are summarized in Table 2.3.

Λ	$C_2(\Lambda)$	$SO(9)$ triplets	$C_2(\lambda)$
(0 0 0 0)	0	$(2\,0\,0\,0) \oplus (0\,0\,1\,0) \oplus (1\,0\,0\,1)$	18
(0 0 0 1)	12	$(3\,0\,0\,0) \oplus (0\,0\,1\,1) \oplus (1\,0\,0\,2)$	30
(1 0 0 0)	18	$(2\,1\,0\,0) \oplus (0\,1\,1\,0) \oplus (1\,1\,0\,1)$	36
(0 0 1 0)	24	$(3\,0\,0\,1) \oplus (0\,0\,2\,0) \oplus (2\,0\,0\,2)$	42
(0 0 0 2)	26	$(4\,0\,0\,0) \oplus (0\,0\,1\,2) \oplus (1\,0\,0\,3)$	44

Table 2.3. Casimir values of F_4 irreps and $SO(9)$ triplets.

2.4 Classification of the lowest lines of the Euler multiplets

Now, we return to analyze the GKRS index (2.2),

$$V_\Lambda \otimes S^+ - V_\Lambda \otimes S^- = \sum_{c\in C} \mathrm{sgn}(c) U_{c\bullet\Lambda}. \tag{2.26}$$

When V_Λ is the one-dimensional irreducible representation of g and it is trivially branched into one-dimensional representation of h. The lowest line of the Euler multiplets of h is equal to a decomposition of the two spinor representations of $SO(\dim g/h)$ into irreducible representations of h. In the case that an Euler number $\mathbf{C}$ is small integer, one can compute the Euler multiplet from $\lambda = c(\Lambda+\rho_g)-\rho_h$ as shown in Ref. [1]. In the case that the Euler number is high integer, e.g. $\mathbf{C} > 10$, the Euler multiplet can be obtained by branching the irreducible representation V_Λ and the spinor representations of $SO(\dim g/h)$ into the representations of h, tensoring them together, and taking the difference between positive and negative spin spaces. In the appendix B, a basic concept of branching from g to h is explained and the branching routine is given in the LiE programing language.

It is observed that when $\dim g/h = 2^n$, $n \in I$, the lowest lines of the Euler multiplets have the right degrees of freedom of the massive and massless supersymmetric representations of the supersymmetric gauge field theories. So, it is convenient to classify the lowest lines of the Euler multipets [4] according to $SO(\dim g/h)$.

1. $SO(4)$:

$$S^+ = 2_s \text{ and } S^- = 2_c.$$

---

[4] The Euler multiplets in this section are written in terms of dimensions of the irreducible representations of h.

- $Sp(4) \supset Sp(2) \times Sp(2)$

$$\begin{aligned} S^+ &= (1,2), \\ S^- &= (2,1). \end{aligned}$$

Since the group $Sp(2)$ is locally isomorphic to the group $SO(3)$, i.e. $Sp(2) \sim SO(3)$, this Euler multiplet corresponds to the $N=1$ smallest massive representation in $3+1$ space-time dimensions when one of the $Sp(2)$ is interpreted as the little group of the inhomogenous Lorentz group $ISO(3,1)$ and to the $N=1$ shortest massless representation in $5+1$ space-time dimensions when one of the $Sp(2) \sim SU(2)$ algebras is interpreted as a chirality group in $SO(4) \sim SU(2)_L \times SU(2)_R$, the light-cone little group of $ISO(5,1)$.

- $SU(3) \supset SU(2) \times U(1)$

$$\begin{aligned} S^+ &= 2_0, \\ S^- &= 1_{1/2} \oplus 1_{-1/2}. \end{aligned}$$

This Euler multiplet corresponds to the $N=2$ massless representation [5] in $3+1$ dimensions when the $U(1) \sim SO(2)$ algebra is interpreted as the little group of $ISO(3,1)$ and to the massless representation in $4+1$ when the $SU(2) \sim SO(3)$ algebra is interpreted as the little group of $ISO(4,1)$.

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[5] The N=2 supersymmetric multiplet with the highest helicity equal to 1/2 is sometimes called hypermultiplet.

It is noticed that $N = 2$ massless representation in 3+1 dimension which is the lowest line of the Euler multiplets of $SU(3) \supset SU(2) \times U(1)$ can be assembled into $N = 1$ massive representation in 3+1 dimensions which is the lowest line of the Euler multiplets of $Sp(4) \supset Sp(2) \times Sp(2)$. On the 4-dimensional coset space, the $N = 2$ massless representation, considered as a fundamental state of one theory at weak (strong) coupling limit, possibly has a duality with $N = 1$ massive representation, considered as a solitonic state of the other theory at strong (weak) coupling limit.

2. $SO(8)$:

$$S^+ = 8_s \text{ and } S^- = 8_c.$$

- $Sp(6) \supset Sp(4) \times Sp(2)$ and $G_2 \supset SU(2) \times SU(2)$

$$\begin{aligned} S^+ &= (5,1) \oplus (1,3), \\ S^- &= (4,2). \end{aligned}$$

  This Euler multiplet corresponds to the $N = 2$ smallest massive representation in $3 + 1$ dimensions and the $N = 2$ shortest massless representation in $5 + 1$ dimensions.

- $SU(5) \supset SU(4) \times U(1)$

$$\begin{aligned} S^+ &= 1_1 \oplus 6_0 \oplus 1_{-1}, \\ S^- &= 4_{1/2} \oplus \overline{4}_{-1/2}. \end{aligned}$$

This Euler multiplet corresponds to the $N = 4$ Yang-Mills massless representation in 3+1 dimensions and, since $SU(4) \sim SO(6)$, a massless representation in $7+1$ dimensions. However, there are two more possibilities to interpret this Euler multiplet, one as a massless representation on the anti-de Sitter space when the $SU(4) \times U(1) \sim SO(6) \times SO(2)$ algebra is interpreted as the subgroup of $SO(6,2)$, the anti-de Sitter group in $6+1$ dimensions, and the other as a massless representation of a conformal theory when interpreted as the conformal group in $5+1$ dimensions.

- $SO(6) \supset SO(4) \times SO(2)$ and $SU(4) \supset SU(2) \times SU(2) \times U(1)$

$$\begin{aligned} S^+ &= (1,1)_1 \oplus (3,1)_0 \oplus (1,3)_0 \oplus (1,1)_{-1}, \\ S^- &= (2,2)_{1/2} \oplus (2,2)_{-1/2}. \end{aligned}$$

There are many possibilities to view this Euler multiplet. It corresponds to the $N = 4$ Yang-Mills supermultiplet in $3+1$ dimensions and to a massless multiplet in either $4+1$ or $5+1$ dimensions.

Note that $N = 4$ massless representation in 3+1 dimensions which is the lowest line of the Euler multiplets of $SU(5) \supset SU(4) \times U(1)$, $SO(6) \supset SO(4) \times SO(2)$ and $SU(4) \supset SU(2) \times SU(2) \times U(1)$ can also be assembled into $N = 2$ massive representation in 3+1 dimensions which is the lowest line of the Euler multiplets of $Sp(6) \supset Sp(4) \times Sp(2)$ and $G_2 \supset SU(2) \times SU(2)$. On the 8-dimensional coset space, the $N = 2$ massless representations, considered as the fundamental states of one theory at weak (strong) coupling limit, possibly have a duality with $N = 1$ massive representations, considered as the solitonic states of the other theory at strong (weak) coupling limit.

3. $SO(16)$:

$$S^+ = 128_s \text{ and } S^- = 128_c.$$

- $F_4 \supset SO(9)$

$$\begin{aligned} S^+ &= 44 \oplus 84, \\ S^- &= 128. \end{aligned}$$

This multiplet can be viewed as the $N = 1$ massive soliton representation in 9+1 dimensions and as the $N = 1$ massless representation in 10+1 dimensions. In the later case, it is called the supergravity triplet where the 44–dimensional representation is the degrees of freedom of the spin-2 graviton, the 84–dimensional is the degrees of freedom of the third-rank antisymmetric tensor field and the 128–dimensional is the degrees of freedom of the Rarita-Schwinger spinor-vector called the gravitino.

- $Sp(10) \supset Sp(8) \times Sp(2)$

$$\begin{aligned} S^+ &= (42,1) \oplus (27,3) \oplus (1,5), \\ S^- &= (48,2) \oplus (8,4). \end{aligned}$$

This Euler multiplet corresponds to the $N = 4$ smallest massive representation in $3+1$ dimensions and the $N = 4$ short massless representation in $5+1$ dimensions.

- $SO(8) \supset SO(4) \times SO(4) \sim SU(2) \times SU(2) \times SU(2) \times SU(2)$

$$\begin{aligned} S^+ &= (5,1,1,1) \oplus (1,5,1,1) \oplus (1,1,5,1) \oplus (1,1,1,5) \\ &\quad \oplus (1,3,3,3) \oplus (3,1,3,3) \oplus (3,3,1,3) \oplus (3,3,3,1), \\ S^- &= (4,2,2,2) \oplus (2,4,2,2) \oplus (2,2,4,2) \oplus (2,2,2,4). \end{aligned}$$

This Euler multiplet corresponds to the $N = 4$ smallest massive representation in $3+1$ dimensions and $N = 2$ massless representaton in 5+1 dimensions.

- $SU(9) \supset SU(8) \times U(1)$

$$\begin{aligned} S^+ &= 1_2 \oplus 28_1 \oplus 70_0 \oplus \overline{28}_{-1} \oplus 1_{-2}, \\ S^- &= 8_{3/2} \oplus 56_{1/2} \oplus \overline{56}_{-1/2} \oplus \overline{8}_{-3/2}. \end{aligned}$$

Since the highest value of $SO(2)$ is 2, the helicity of graviton in 3+1 dimensions, this Euler multiplet corresponds to the $N = 8$ supergravity representation.

- $SO(10) \supset SO(8) \times SO(2)$

$$\begin{aligned} S^+ &= 1_2 \oplus 28_1 \oplus (35_s)_0 \oplus (35_c)_0 \oplus 28_{-1} \oplus 1_{-2}, \\ S^- &= 8_{3/2} \oplus (56_v)_{1/2} \oplus (56_v)_{-1/2} \oplus 8_{-3/2}. \end{aligned}$$

This Euler multiplet corresponds to the $N = 8$ supergravity representation in $3+1$ dimensions. After using $SO(8)$ triality, it also corresponds to the chiral $N = 4$ supermultiplet of type IIB string theory in $9+1$ dimensions.

- $SU(6) \supset SU(4) \times SU(2) \times U(1) \sim SO(6) \times SO(3) \times SO(2)$

$$\begin{aligned} S^+ &= (1,1)_2 \oplus (10,1)_1 \oplus (6,3)_1 \oplus (20',1)_0 \oplus (15,3)_0 \\ &\oplus (1,5)_0 \oplus (\overline{10},1)_{-1} \oplus (6,3)_{-1} \oplus (1,1)_{-2}, \\ S^- &= (4,2)_{3/2} \oplus (20,2)_{1/2} \oplus (\overline{4},4)_{1/2} \oplus (\overline{20},2)_{-1/2} \\ &\oplus (4,4)_{-1/2} \oplus (\overline{4},2)_{-3/2}. \end{aligned}$$

This Euler multiplet also corresponds to the $N = 8$ supergravity representation in $3+1$ dimensions and to a massless representation in $4+1$ and $7+1$ dimensions.

Note that $N = 8$ supergravity multiplet in 3+1 dimensions which is the lowest line of the Euler multiplets of $SU(9) \supset SU(8) \times U(1)$ and $SO(10) \supset SO(8) \times SO(2)$ can also be assembled into $N = 2$ massive representation in 3+1 dimensions which is the lowest line of the Euler multiplets of $Sp(10) \supset Sp(8) \times Sp(2)$. On the 16-dimensional coset space, the $N = 8$ massless representations, considered as the fundamental states of one theory at weak (strong) coupling limit, possibly have a duality with $N = 4$ massive representations, considered as the solitonic states of the other theory at strong (weak) coupling limit.

4. $SO(32)$:

- $E_6 \supset SO(10) \times SO(2)$

$$\begin{aligned} S^+ &= 1_4 \oplus 120_3 \oplus 770_2 \oplus 1050_2 \oplus 3696_1 \\ &\oplus 4312_1 \oplus 660_0 \oplus 4125_0 \oplus 8085_0 \oplus 4312_{-1} \\ &\oplus 3696_{-1} \oplus 1050_{-2} \oplus 770_{-2} \oplus 120_{-3} \oplus 1_{-4}, \\ S^- &= (16_s)_{7/2} \oplus (560_c)_{5/2} \oplus (672_c)_{3/2} \oplus (3696_c)_{3/2} \\ &\oplus (2640_s)_{1/2} \oplus (8800_c)_{1/2} \oplus (8800_s)_{-1/2} \oplus (2640_c)_{-1/2} \\ &\oplus (672_s)_{-3/2} \oplus (3696_s)_{-3/2} \oplus (560_c)_{-5/2} \oplus (16_c)_{-7/2}. \end{aligned}$$

  This Euler multiplet possibly corresponds to a massive supermultiplet with spin > 2 in $10+1$ dimensions.

There also exist the lowest lines of the Euler multiplets that are of $SO(6)$, $SO(10)$, $SO(12)$, etc. and not all of them can be interpreted as the degrees of freedom of the supersymmetric gauge field theories except $Sp(8) \supset Sp(6) \times Sp(2)$ that its lowest line emerges as $N = 3$ massive (massless) representation in 3+1 (5+1) dimensions.

In conclusion, from the facts of group theory, there are different coset spaces. For the 4-, 8-, and 16-dimensional coset spaces, there exist the lowest lines of the Euler multiplets corresponding to the massive and massless supermultiplets.

Before closing this chapter, we would like to mention the existence of $SO(16)$ triplets [9] that cannot be explained by the GKRS construction.

2.5 The $SO(16)$ triplets

$SO(16)$ is a semisimple Lie algebra of rank 8. The equal rank embedding of $SO(16)$ into E_8 by the GKRS construction generates the Euler multiplets that have 135 irreducible representations of $SO(16)$. Since the coset space of $E_8/SO(16)$ is 128-dimensional, the dimension of the lowest line of the Euler multiplets is equal to 2^{64}, the dimension of the sum of the two spinor representations of $SO(128)$. The lowest line of the $E_8/SO(16)$ emerges when the quadratic Casimir value of $SO(16)$ irreps is equal to

$$|\rho_{E_8}|^2 - |\rho_{SO(16)}|^2 = 480. \tag{2.27}$$

However, when we look at an equality of quadratic Casimir value of $SO(16)$, there also exists an abundance of $SO(16)$ triplets similar to the $SO(9)$. Since $SO(16)$ has two spinors of the same dimension, we give some example of triplets associated with one of them in Table 2.4, 2.5 and 2.6. An irreducible representation of $SO(16)$ in the triplets is given in the ω-basis and is characterized by generalized Dynkin indices [10]

$$I_p = \sum_{\text{rep}} w^p, \qquad p = 0, 2, 4, 6, 8, 10, 12, 14, \tag{2.28}$$

where w's are the weights in the irreducible representation of $SO(16)$. The quadratic Casimir value of the $SO(16)$ irrep is related to the dimension of the irrep I_0 and the second order Dynkin index I_2 by

$$C_2 = \frac{120 I_2}{8 I_0}. \tag{2.29}$$

Note that in each triplet only the generalized Dynkin indices I_0, I_2, and I_6 of the first irrep is equal to the sum of those of the second and third irreps. This property of a $SO(16)$ triplet is the same as that of the $SO(9)$ triplet, but the dimension of the lowest line of $SO(16)$ triplet is not equal to 2^n, the dimension of the spinor module of $SO(n)$. Hence, the triplets that exist in $SO(16)$ may be an exception to GKRS construction. Nevertheless, a pattern of $SO(16)$ triplets possibly has some connection to that of $SO(9)$ triplets.

- The first triplet:

  Each irrep in the triplet has the quadratic Casimir value equal to 60.

Irrep	(01000001)	(02000000)	(00000100)
I_0	13312	5304	8008
I_2	53248	21216	32032
I_4	241664	98752	142912
I_6	1196032	508800	687232
I_8	6275072	2821888	3496192
I_{10}	34275328	16559616	18575872
I_{12}	192610304	101727232	102065152
I_{14}	1104412672	649574400	575260672

Table 2.4. The first $SO(16)$ triplet and its Dynkin indices.

- The second triplet:

  Each irrep has the quadratic Casimir value equal to 64.

Irrep	(20000001)	(00000002)	(21000000)
I_0	15360	6435	8925
I_2	65536	27456	38080
I_4	323584	130944	192640
I_6	1785856	676608	1109248
I_8	10731520	3724800	7135744
I_{10}	68878336	21605376	50713600
I_{12}	465240064	130996224	393594880
I_{14}	3268673536	825274368	3286355968

Table 2.5. The second $SO(16)$ triplet and its Dynkin indices.

- The third triplet:

  Each irrep has the quadratic Casimir value equal to 72.

Irrep	(10001000)	(40000000)	(00100001)
I_0	60060	3740	56320
I_2	288288	17952	270336
I_4	1591744	104896	1486848
I_6	9717120	722304	8994816
I_8	63847168	5817088	58245120
I_{10}	442541568	54584832	396558336
I_{12}	3187846144	589597696	2803826688
I_{14}	23608596480	7166367744	20402405376

Table 2.6. The third $SO(16)$ triplet and its Dynkin indices.

CHAPTER 3
GKRS CONSTRUCTION OF AN EULER MULTIPLET: BASIC LIE SUPERALGEBRAS

In this chapter, we apply the GKRS construction to basic Lie superalgebras. The Kac character formulas are derived in terms of equal rank subalgebras in section 3.1. We give a simple and explicit formulation for a typical representation of type I Lie superalgebras in section 3.2 and build an Euler multiplet of type I Lie superalgebras from that of reductive Lie algebras in section 3.3. Kostant's cubic Dirac operator is constructed for full Lie superalgebras and then for equal rank embeddings in section 3.4.

3.1 Kac character formulas of basic Lie superalgebras

Now, we extend the results of semisimple complex Lie algebras to basic Lie superalgebras g with a non-degenerate Killing form $\mathcal{K} : g \times g \rightarrow \mathbb{C}$. According to Kac's classification [11], there are two types of basic Lie superalgebras, type I which is $su(m|n)$ and $osp(2|2n)$ and type II which is $osp(2m+1|2n)$, $osp(1|2n)$, $osp(2m|2n)$, $osp(4|2;\alpha)$, $F(4)$, and $G(3)$.

Let $g = g_{even} \oplus g_{odd}$ be the Lie superalgebras with the root system $\Phi = \Phi_{even} \cup \Phi_{odd}$. For type I, g_{even} is simple, i.e. $g_{even} = g_0$, and, for type II, g_{even} can be graded into $g_2 \oplus g_0 \oplus g_{-2}$. While, for the odd part of both type I and II, odd generators can be graded into fermionic creation and annihilation ones, i.e. $g_{odd} = g_1 \oplus g_{-1}$. The Poincaré-Birkhoff-Witt theorem for Lie algebras can be applied to the case of Lie superalgebras with some extension [12]. This grading gives us a universal enveloping algebra, $\mathcal{U}(g)$, e.g., for type I,

$$\mathcal{U}(g) = \mathcal{U}(g_1) \otimes \mathcal{U}(g_0) \otimes \mathcal{U}(g_{-1}).$$

Define root subsystems, $\overline{\Phi}_{even}$ and $\overline{\Phi}_{odd}$, such that $\overline{\Phi}_{even} = \{\alpha \mid \alpha/2 \notin \Phi_{odd}\}$ and $\overline{\Phi}_{odd} = \{\beta \mid 2\beta \notin \Phi_{even}\}$. Φ_{even}, Φ_{odd}, $\overline{\Phi}_{even}$ and $\overline{\Phi}_{odd}$ are invariant under the action of the Weyl group of g_{even}. Hence, the Weyl group of g is equal to that of g_{even}, i.e. $W(g) = W(g_{even})$. Define the Weyl vector of g to be one-half the sum of positive even roots minus one-half the sum of positive odd roots, i.e. $\rho = \rho_{even} - \rho_{odd}$. Let $V(\Lambda)$ be an irreducible representation of g with Λ as a highest weight in the dual Cartan subalgebra. The highest weight representations of g are classified into typical and atypical. The representation is typical if, for $\forall \beta \in \overline{\Phi}^{+}_{odd}$, $(\Lambda + \rho, \beta) \neq 0$; otherwise, it is atypical. The typical Kac character and supercharacter of $V(\Lambda)$ are defined, respectively, as

$$\mathrm{ch}V(\Lambda) = \frac{N_1}{N_0} \sum_{w \in W(g)} \mathrm{sgn}(w) e^{w(\Lambda+\rho)}, \tag{3.1}$$

$$\mathrm{sch}V(\Lambda) = \frac{N_1'}{N_0} \sum_{w \in W(g)} \mathrm{sgn}(\overline{w}) e^{w(\Lambda+\rho)}, \tag{3.2}$$

where

$$N_0 = \prod_{\alpha \in \Phi^{+}_{even}} \left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right),$$

$$N_1 = \prod_{\beta_+ \in \Phi^{+}_{odd}} \left(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\right),$$

and

$$N_1' = \prod_{\beta_+ \in \Phi^+_{odd}} \left(e^{\frac{\beta}{2}} - e^{-\frac{\beta}{2}}\right).$$

The $\mathrm{sgn}(w)$ and $\mathrm{sgn}(\overline{w})$ are sign change due to number of reflections with respect to Φ^+_{even} and $\overline{\Phi}^+_{even}$.

In general, in an equal rank embedding of Lie superalgebras r in g with $\Phi^+(r) \subset \Phi^+(g)$ and $\mathbf{C}$ as the index of the Weyl group of r in the Weyl group of g, a $\mathbf{C}$-multiplet of r is obtained by

$$c \bullet \Lambda := c(\Lambda + \rho)_g - \rho_r,$$

where $c \in C$. Under the condition that $\Phi^+_{even,odd}(r)$ is invariant under the action of c, i.e. $c \cdot \Phi^+_{even,odd}(r) = \Phi^+_{even,odd}(r)$, the typical Kac character formula of g-module $V(\Lambda)$ can be written in terms of r-module $U(c \bullet \Lambda)$ as follows:

$$\begin{aligned} \mathrm{ch}V(\Lambda) &= \frac{\prod_{\beta \in \Phi^+_{odd}(g)}(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}})}{\prod_{\alpha \in \Phi^+_{even}(g)}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})} \sum_{w \in W(g)} \mathrm{sgn}(w) e^{w(\Lambda + \rho_g)} \\ &= \left(\frac{\prod_{\beta \in \Phi^+_{odd}(g/r)}(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}})}{\prod_{\alpha \in \Phi^+_{even}(g/r)}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})} \right) \sum_{c \in C} \mathrm{sgn}(c) \mathrm{ch}U(c \bullet \Lambda). \end{aligned} \tag{3.3}$$

Similarly done, the typical Kac supercharacter becomes

$$\mathrm{sch}V(\Lambda) = \left(\frac{\prod_{\beta \in \Phi^+_{odd}(g/r)}(e^{\frac{\beta}{2}} - e^{-\frac{\beta}{2}})}{\prod_{\alpha \in \Phi^+_{even}(g/r)}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})} \right) \sum_{c \in C} \mathrm{sgn}(c) \mathrm{sch}U(c \bullet \Lambda). \tag{3.4}$$

For type I Lie superalgebras, there are both typical and atypical representations. For the typical representation, superdimension, sdim $V(\Lambda) =$ dim $V_{even}(\Lambda) -$

dim $V_{odd}(\Lambda)$, is equal to zero. Whereas, sdim $V(\Lambda)$ of the atypical representation is not. Every type I odd root is zero-length and $\Phi^+_{even} = \overline{\Phi}^+_{even}$. Since ρ_{odd} is invariant under the action of the Weyl group, i.e. $w\rho_{odd} = \rho_{odd}$. The type I typical Kac character formula (3.1) can be written as

$$\begin{aligned} \mathrm{ch}V(\Lambda) &= \prod_{\beta_+\in\Phi_1^+}(1+e^{-\beta_+})\mathrm{ch}V_0(\Lambda) \\ &= \prod_{\beta_-\in\Phi_1^-}(1+e^{\beta_-})\mathrm{ch}V_0(\Lambda). \end{aligned} \tag{3.5}$$

Multiplying out the product factor on the rhs of (3.5), we obtain a Chern character of an exterior algebra over g_{-1},

$$\prod_{\beta_-\in\Phi_1^-}(1+e^{\beta_-}) = \sum_{n=0}^{n=dim(\Phi_{-1}^-)} \mathrm{ch}(\wedge^n g_{-1}) = \mathrm{ch}(\wedge g_{-1}).$$

So, (3.5) simply becomes

$$\mathrm{ch}V(\Lambda) = \mathrm{ch}(\wedge g_{-1})\mathrm{ch}V_0(\Lambda). \tag{3.6}$$

Similarly, the type I typical Kac supercharacter can be shown to be

$$\begin{aligned} \mathrm{sch}V(\Lambda) &= \prod_{\beta_-\in\Phi_1^-}(1-e^{\beta_-})\mathrm{ch}V_0(\Lambda) \\ &= \mathrm{sch}(\wedge g_{-1})\mathrm{ch}V_0(\Lambda). \end{aligned} \tag{3.7}$$

On the other hand, since $\{g_{-1}, g_{-1}\} = 0$, the universal enveloping algebra over g_{-1}, $\mathcal{U}(g_{-1})$, is isomorphic to the exterior algebra over g_{-1}, $\wedge(g_{-1})$. The g-module $V(\Lambda)$ can be induced by applying the antisymmetric combinations of the g_{-1} generators on $V_0(\Lambda)$, i.e.

$$V(\Lambda) = \wedge(g_{-1}) \otimes V_0(\Lambda) \simeq \mathcal{U}(g_{-1}) \otimes V_0(\Lambda). \tag{3.8}$$

The character and supercharacter of (3.8) are exactly (3.6) and (3.7).

In an equal rank embedding, $g \supset r$, of type I which has $\mathbf{C}$ as the index of $W(r)$ in $W(g)$ and is restricted to the condition that $\Phi^+(g) \supset \Phi^+(r)$, (3.5) becomes

$$\mathrm{ch}V(\Lambda) = \frac{1}{\Delta} \prod_{\beta_- \in \Phi_{\bar{1}}^-(g/r)} (1 + e^{\beta_-}) \sum_{c \in C} \mathrm{sgn}(c) \left(\prod_{\beta_- \in \Phi_{\bar{1}}^-(r)} (1 + e^{\beta_-}) \mathrm{ch}U_0(c \bullet \Lambda) \right),$$

i.e.

$$\mathrm{ch}V(\Lambda) \left(\mathrm{ch}\ S^+ - \mathrm{ch}\ S^-\right) = \mathrm{ch}\left(\wedge(g_{-1}/r_{-1})\right) \sum_{c \in C} \mathrm{sgn}(c) \mathrm{ch}U(c \bullet \Lambda). \tag{3.9}$$

Similarly done for the supercharacter, we obtain

$$\mathrm{sch}V(\Lambda) \left(\mathrm{ch}\ S^+ - \mathrm{ch}\ S^-\right) = \mathrm{sch}\left(\wedge(g_{-1}/r_{-1})\right) \sum_{c \in C} \mathrm{sgn}(c) \mathrm{sch}U(c \bullet \Lambda). \tag{3.10}$$

(3.9) and (3.10) correspond to

$$V(\Lambda) \otimes S^+ - V(\Lambda) \otimes S^- = \wedge(g_{-1}/r_{-1}) \otimes \sum_{c \in C} \mathrm{sgn}(c) U(c \bullet \Lambda). \tag{3.11}$$

Now, (3.11) can also be derived explicitly from (3.8). By decomposing the g_{-1} basis such that $g_{-1} = (g_{-1}/r_{-1}) \oplus r_{-1}$, there exists a map $(g_{-1}/r_{-1}) \oplus r_{-1} \rightarrow$

$(g_{-1}/r_{-1}) \otimes 1 + 1 \otimes r_{-1}$ from $(g_{-1}/r_{-1}) \oplus r_{-1}$ to $(g_{-1}/r_{-1}) \otimes r_{-1}$ which extends uniquely to an isomorphism of exterior algebra,

$$\wedge\,(g_{-1}/r_{-1} \oplus r_{-1}) \simeq \wedge(g_{-1}/r_{-1}) \otimes \wedge(r_{-1}).$$

By substituting the above equation into (3.8) and tensoring on both side by $(S^+ - S^-)$, we obtain

$$\begin{aligned} V(\Lambda) \otimes S^+ - V(\Lambda) \otimes S^- &= \wedge(g_{-1}/r_{-1}) \otimes \sum_{c \in C} \operatorname{sgn}(c)\,(\wedge(r_{-1}) \otimes U_0(c \bullet \Lambda)) \\ &= \wedge(g_{-1}/r_{-1}) \otimes \sum_{c \in C} \operatorname{sgn}(c) U(c \bullet \Lambda), \end{aligned}$$

which is exactly (3.11).

For *osp*$(1|2n)$ of type II Lie superalgebras, every *osp*$(1|2n)$ odd root has length equal to one-half of the short positive one and

$$\rho_{osp(1|2n)} = \rho_{even} - \rho_{odd} = \left(n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2}\right).$$

Furthermore, every *osp*$(1|2n)$ representation is typical, but dim $V_{even}(\Lambda) \neq$ dim $V_{odd}(\Lambda)$. Nevertheless, the Kac character and supercharacter formulas of a *osp*$(1|2n)$ representation can be shown to be similar to that of Lie algebra, i.e.

$$\operatorname{ch}V(\Lambda) = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\Lambda+\rho)}}{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho)}}, \tag{3.12}$$

$$\operatorname{sch}V(\Lambda) = \frac{\sum_{w \in W} \operatorname{sgn}(\overline{w}) e^{w(\Lambda+\rho)}}{\sum_{w \in W} \operatorname{sgn}(\overline{w}) e^{w(\rho)}}. \tag{3.13}$$

The equal rank embedding of $osp(1|2m) \times osp(1|2n-2m)$ in $osp(1|2n)$ is the only possible type with the full subsuperalgebra. In this case, all odd generators of g are completely eaten by the r. Whereas, the even generators of g which are left form the orthogonal complement basis to the even basis of r under the killing form of g. Under the restriction to r-module, (3.12) and (3.13) simply become

$$\mathrm{ch}V(\Lambda) = \frac{1}{\Delta}\sum_{c\in C}\mathrm{sgn}(c)\mathrm{ch}U(c\bullet\Lambda), \tag{3.14}$$

and

$$\mathrm{sch}V(\Lambda) = \frac{1}{\Delta}\sum_{c\in C}\mathrm{sgn}(c)\mathrm{sch}U(c\bullet\Lambda). \tag{3.15}$$

Both (3.14) and (3.15) correspond to

$$V(\Lambda)\otimes S^{+} - V(\Lambda)\otimes S^{-} = \sum_{c\in C}\mathrm{sgn}(c)U(c\bullet\Lambda). \tag{3.16}$$

For the rest of type II, we use Kac character and supercharacter formulas of equal rank embedding, (3.3) and (3.4). Let g_1 and g_{-1} be N-dimensional vector spaces of generators. There is a canonical linear map from $\wedge^a g_1 \otimes \wedge^b g_{-1}$ to $\wedge^{a+b}(g_1 \oplus g_{-1})$, which takes $((g_1)_1 \wedge \cdots \wedge (g_1)_a) \otimes ((g_{-1})_1 \wedge \cdots \wedge (g_{-1})_b)$ to $((g_1)_1 \wedge \cdots \wedge (g_1)_a \wedge (g_{-1})_1 \wedge \cdots \wedge (g_{-1})_b)$. This determines a linear isomorphism

$$\begin{aligned}\wedge(g_1\oplus g_{-1}) &\simeq \bigoplus_{a=0}^{N}\left(\wedge^{N-a}g_1\otimes\wedge^a g_{-1}\right)\\ &\simeq \wedge g_1\otimes\wedge g_{-1}.\end{aligned}$$

The prefactor N_1 of Kac character formula is the character of the exterior algebra over the direct sum of g_1 and g_{-1} vector spaces

$$\begin{aligned}\prod_{\beta\in\Phi_1^+}\left(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}}\right) &= \prod_{\beta_\pm\in\Phi_1^\pm}\left(e^{\frac{\beta_+}{2}}+e^{\frac{\beta_-}{2}}\right)\\ &= \mathrm{ch}\wedge(g_1\oplus g_{-1}).\end{aligned}$$

Finally, the type II typical Kac character and supercharacter can be written as

$$\mathrm{ch}V(\Lambda)\left(\mathrm{ch}\ S^+ - \mathrm{ch}\ S^-\right) = \mathrm{ch}\wedge(g_1/r_1\oplus g_{-1}/r_{-1})\sum_{c\in C}\mathrm{sgn}(c)\mathrm{ch}U(c\bullet\Lambda), \quad (3.17)$$

and

$$\mathrm{sch}V(\Lambda)\left(\mathrm{ch}\ S^+ - \mathrm{ch}\ S^-\right) = \mathrm{sch}\wedge(g_1/r_1\oplus g_{-1}/r_{-1})\sum_{c\in C}\mathrm{sgn}(c)\mathrm{sch}U(c\bullet\Lambda), \quad (3.18)$$

which correspond to

$$V(\Lambda)\otimes S^+ - V(\Lambda)\otimes S^- = \wedge(g_1/r_1\oplus g_{-1}/r_{-1})\otimes\sum_{c\in C}\mathrm{sgn}(c)U(c\bullet\Lambda). \quad (3.19)$$

3.2 Representations of $\wedge g_{-1}$ of type I Lie superalgebras

For type I Lie superalgebras with non-degenerate Killing form, the typical representation is induced by applying g_{-1} generators on g_0-module. Therefore, we need to know an explicit representation of $\wedge g_{-1}$ in terms of g_0-module. Let N be dimension of Φ_1^-. The exterior algebra over g_{-1} is

$$\wedge g_{-1} = \bigoplus_{k=0}^{N} \wedge^k g_{-1},$$

with dimension,

$$\dim(\wedge g_{-1}) = \bigoplus_{k=0}^{N} \dim(\wedge^k g_{-1}) = \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} = 2^N,$$

and superdimension,

$$\mathrm{sdim}(\wedge g_{-1}) = \bigoplus_{k=0}^{N} \mathrm{sdim}(\wedge^k g_{-1}) = \sum_{k=0}^{N} (-1)^k \begin{pmatrix} N \\ k \end{pmatrix} = 0.$$

Recall that $\{g_{-1}, g_{-1}\} = 0$. Let $Q^\dagger_{i=1,2,\ldots,N}$ be N completely antisymmetric fermionic generators of g_{-1} which transform in the fundamental representation of g_0. The fermionic generators generate even and odd modules which are isomorphic to a direct sum of two spinor representations of $so(2N)$. One of the spinor representations of $so(2N)$ is the even module of $\wedge g_{-1}$, called bosonic module, and the other is the odd module of $\wedge g_{-1}$, called fermionic module. Let T^+ be the bosonic module and T^- be the fermionic module of $so(2N)$ such that

$$\begin{aligned} T^+ &= \wedge^0 g_{-1} \oplus \wedge^2 g_{-1} \oplus \wedge^4 g_{-1} \oplus \ldots \\ &= \left((Q_i^\dagger)^0 \oplus (Q_i^\dagger)^2 \oplus (Q_i^\dagger)^4 \ \oplus \ldots \right) |1>_0 \\ &\equiv \text{bosonic module,} \end{aligned}$$

and

$$\begin{aligned} T^- &= \wedge^1 g_{-1} \oplus \wedge^3 g_{-1} \oplus \wedge^5 g_{-1} \oplus \dots \\ &= \Big((Q_i^\dagger)^1 \oplus (Q_i^\dagger)^3 \oplus (Q_i^\dagger)^5 \ \oplus \dots\Big) |1>_0 \\ &\equiv \text{fermionic module.} \end{aligned}$$

With restriction to g_0-module, the type I typical representation is

$$\begin{aligned} V(\Lambda) &= \wedge g_{-1} \otimes V_0(\Lambda) \\ &= \Big(T^+ \oplus T^-\Big) \otimes V_0(\Lambda). \end{aligned}$$

For $su(m|n)$, $\wedge g_{-1}$ representation is isomorphic to a direct sum of two spinor representations of $so(2mn)$. With restriction to $su(m) \times su(n) \times u(1)$, $Q_i^\dagger$ transforms as $(m,n)_{-1}$, where -1 is a $u(1)$ charge.

Ex.1 $su(2|1)$

$$\dim(g_{-1}) = 2^2 = 1 + 2 + 1$$

$Q_i^\dagger \sim 2_{-1}$

$$\begin{aligned} T^+ &= 1_0 \oplus 1_{-2} \\ T^- &= 2_{-1} \end{aligned}$$

$$V_{su(2|1)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{su(2)\times u(1)}(\Lambda)$$

Ex.2 $su(3|1)$

$$\dim(g_{-1}) = 2^3 = 1 + 3 + 3 + 1$$

$Q_i^\dagger \sim 3_{-1}$

$$\begin{aligned} T^+ &= 1_0 \oplus \overline{3}_{-2} \\ T^- &= 3_{-1} \oplus 1_{-3} \end{aligned}$$

$$V_{su(3|1)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{su(3)\times u(1)}(\Lambda)$$

Ex.3 $su(3|2)$

$$\dim(g_{-1}) = 2^6 = 1 + 6 + 15 + 20 + 15 + 6 + 1$$

$Q_i^\dagger \sim (3,2)_{-1}$

$$\begin{aligned} T^+ &= (1,1)_0 \oplus (\overline{3},3)_{-2} \oplus (6,1)_{-2} \oplus (6,1)_{-4} \oplus (3,3)_{-4} \oplus (1,1)_{-6} \\ T^- &= (3,2)_{-1} \oplus (8,2)_{-3} \oplus (4,1)_{-3} \oplus (\overline{3},2)_{-5} \end{aligned}$$

$$V_{su(3|2)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{su(3)\times su(2)\times u(1)}(\Lambda)$$

Ex.4 $su(4|2)$

$$\dim(g_{-1}) = 2^8 = 1+8+28+56+70+56+28+8+1$$

$Q_i^\dagger \sim (4,2)_{-1}$

$$\begin{aligned} T^+ &= (1,1)_0 \oplus (10,1)_{-2} \oplus (6,3)_{-2} \oplus (20',1)_{-4} \oplus (15,3)_{-4} \\ &\quad \oplus (1,5)_{-4} \oplus (\overline{10},1)_{-6} \oplus (6,3)_{-6} \oplus (1,1)_{-8} \\ T^- &= (4,2)_{-1} \oplus (20,2)_{-3} \oplus (\bar{4},4)_{-3} \oplus (\overline{20},2)_{-5} \oplus (4,4)_{-5} \oplus (\bar{4},2)_{-7} \end{aligned}$$

$$V_{su(4|2)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{su(4)\times su(2)\times u(1)}(\Lambda)$$

For $osp(2|2n)$, $\wedge g_{-1}$ representation is a direct sum of two spinor representations of $so(4n)$. With restriction to $sp(2n)\times u(1)$, $Q_i^\dagger$ transforms as $(2n)_{-1}$ with -1 as a $u(1)$ charge.

Ex.5 $osp(2|4)$

$$\dim(g_{-1}) = 2^4 = 1+4+6+4+1$$

$Q_i^\dagger \sim 4_{-1}$

$$\begin{aligned} T^+ &= 1_0 \oplus 5_{-2} \oplus 1_{-2} \oplus 1_{-4} \\ T^- &= 4_{-1} \oplus 4_{-3} \end{aligned}$$

$$V_{osp(2|4)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{sp(4)\times u(1)}(\Lambda)$$

Ex.6 $osp(2|6)$

$$\dim(g_{-1}) = 2^6 = 1 + 6 + 15 + 20 + 15 + 6 + 1$$

$Q_i^\dagger \sim 6_{-1}$

$$\begin{aligned} T^+ &= 1_0 \oplus 14_{-2} \oplus 1_{-2} \oplus 14_{-4} \oplus 1_{-4} \oplus 1_{-6} \\ T^- &= 6_{-1} \oplus 14'_{-3} \oplus 6_{-3} \oplus 6_{-5} \end{aligned}$$

$$V_{osp(2|6)}(\Lambda) = (T^+ \oplus T^-) \otimes V_{sp(6)\times u(1)}(\Lambda)$$

3.3 Building type I Euler multiplets

For type I Lie superalgebras, there is a remarkable point we would like to mention. From the multiplet spectrum of equal rank embedding of reductive Lie algebras, $r_0 \subset g_0$, we can build the spectrum of the type I Lie superalgebras on top of them by simply tensoring them with a module generated by r_{-1} generators. The r_{-1} generators transform in the fundamental representation of r_0.

Define $\wedge(r_{-1}) = R^+ \oplus R^-$ such that

$$U(\Lambda) = \wedge(r_{-1}) \otimes U_0(\Lambda) = (R^+ \oplus R^-) \otimes U_0(\Lambda).$$

Tensoring on both side of (2.2) with $\wedge(r_{-1})$, we obtain

$$\begin{aligned} \wedge(r_{-1}) \otimes \left(V_0(\Lambda) \otimes S^+ - V_0(\Lambda) \otimes S^-\right) &= \sum_{c\in C} \mathrm{sgn}(c)\,(\wedge(r_{-1}) \otimes U_0(c \bullet \Lambda)) \\ &= \sum_{c\in C} \mathrm{sgn}(c) U(c \bullet \Lambda). \qquad (3.20) \end{aligned}$$

Under restriction to r_0-module, whenever $V_0(\Lambda)$ on the lhs of (3.20) is $su(n) \times u(1)$, or $sp(2n) \times u(1)$, there is an emergence of type I typical **C**-multiplet on the rhs. Notice in the case that $U(c \bullet \Lambda)$ is $su(m|n)$, the representations of $R^{\pm}$ are similar to those of $S^{\pm}$ except $u(1)$ values.

Let $g = r \oplus p$ be Lie superalgebras where p is the orthogonal complement to r with respect to the Killing form of g. Let $p = p_{even} \oplus p_{odd} = p_0 \oplus p_1 \oplus p_{-1}$. Tensoring on both sides of (3.20) by $\wedge p_{-1}$, we get

$$V(\Lambda) \otimes S^{+} - V(\Lambda) \otimes S^{-} = \wedge p_{-1} \otimes \sum_{c \in C} \mathrm{sgn}(c) U(c \bullet \Lambda), \tag{3.21}$$

which is exactly (3.11). Now, we need to get rid of $\wedge p_{-1}$ on the rhs of (3.21) by mapping it into identity. To do so, we do a tensor product on both sides of (3.21) by the following contraction, sometimes called an internal product, on exterior powers between vector space and its dual:

$$\wedge^{N}(p_{odd}) = \wedge^{N} p_1 \otimes \wedge^{N} p_{-1} = \mathbf{1}.$$

(3.21) becomes

$$(\wedge^{N} p_{odd}) \otimes V(\Lambda) \otimes (S^{+} - S^{-}) = \mathbf{1} \otimes \sum_{c \in C} \mathrm{sgn}(c) U(c \bullet \Lambda). \tag{3.22}$$

As already known, the character of a Dirac operator, $\not{\partial} \in C(p)$, with a map $\not{\partial} : S^{\pm} \to S^{\mp}$ is

$$\mathrm{ch}(\not{\partial}) = \mathrm{ch}(S^{+}) - \mathrm{ch}(S^{-}).$$

The character of (2.1) implies that there exists a Kostant's Dirac operator, $\not{\mathcal{K}} \in \mathcal{U}(g_{even}) \otimes C(p_{even})$, with a map

$$\not{\mathcal{K}}_\lambda : V_\lambda \otimes S^\pm \rightarrow V_\lambda \otimes S^\mp. \tag{3.23}$$

Similarly, the character of (3.22) suggests that there exists the operator for Lie superalgebras, $\not{\mathcal{K}}_\Lambda \in \mathcal{U}(p_{odd}) \otimes \mathcal{U}(g_{even} \oplus g_{odd}) \otimes C(p_{even})$, with a map

$$\not{\mathcal{K}}_\Lambda : V(\Lambda) \otimes S^\pm \rightarrow V(\Lambda) \otimes S^\mp. \tag{3.24}$$

3.4 Kostant's cubic Dirac operator of Lie superalgebras

Let L_i and F_a be even and odd generators of Lie superalgebras such that

$$[L_i, L_j] = f_{[ijk]} L_k,$$

$$\{F_a, F_b\} = f_{i(ab)} L_i,$$

and

$$[L_i, F_a] = f_{[ia]b} F_b,$$

where i,j, k = 1, 2, ..., $\dim(g_0)$ and a, b = 1, 2, ..., $\dim(g_1)$. The Kostant's cubic Dirac operator of Lie superalgebras is extended from that of Lie algebras by adding just two terms, a linear term in odd operators and a structure constant term, i.e.

$$\not{\mathcal{K}}_\Lambda = \not{\mathcal{K}}_\Lambda^0 + \not{\mathcal{K}}_\Lambda^1, \tag{3.25}$$

where

$$\not{\mathcal{K}}_\Lambda^0 = \gamma_i L_i - \frac{1}{2}\gamma_{[ijk]} f_{[ijk]} \tag{3.26}$$

and

$$\not{\mathcal{K}}_\Lambda^1 = \alpha_a F_a - \frac{1}{2}\gamma_i \alpha_{(ab)} f_{i(ab)} \tag{3.27}$$

The sum over all indices is assumed in the above equations, where $[\ldots]$ in the subscript is for antisymmetric sum and $(\ldots)$ for symmetric sum. (3.26) is the Kostant's cubic Dirac operator for reductive Lie algebras. The γ-matrices associated to even generators are normalized so that

$$\{\gamma_i, \gamma_j\} = \delta_{ij},$$

which gives

$$\{\gamma_{i'}, \gamma_{[ijk]}\} = \delta_{i'k}\gamma_{[ij]}.$$

The α-matrices associated to odd generators are subjected to the following conditions.

(I) $\{\gamma_i, \alpha_a\} = 0$. This relation is consistent with the antisymmetric property of product of even and odd generators.

(II) $\{\alpha_a, [\alpha_b, \alpha_c]\} + \{[\alpha_a, \alpha_c], \alpha_b\} = [\alpha_{(ab)}, \alpha_c] = 0$. This property is due to invariance of odd gennerators under Killing form.

Squaring (3.26), we get

$$\begin{aligned}(\not{\mathcal{K}}_\Lambda^0)^2 &= \frac{1}{2}\gamma_{(ij)}\{L_i, L_j\} + \frac{1}{2}\gamma_{[ij]}[L_i, L_j] \\ &\quad -\frac{1}{2}\{\gamma_{i'}, \gamma_{[ijk]}\} f_{[ijk]} L_{i'} + \left(\frac{1}{2}\gamma_{[ijk]} f_{[ijk]}\right)^2\end{aligned}$$

$$= \frac{1}{2}\gamma_{(ij)}\{L_i, L_j\} + \left(\frac{1}{2}\gamma_{[ijk]}f_{[ijk]}\right)^2 \tag{3.28}$$

Notice that the linear terms in even generators cancel each other. Thus, (3.28) is explicity invaraint under the action of even and odd generators. The first term of (3.28) is the quadratic Casimir operator of reductive Lie algebras,

$$C_2^0(\Lambda) = \frac{1}{2}\gamma_{(ij)}\{L_i, L_j\}. \tag{3.29}$$

Since ρ_0, one-half the sum of positive even roots, can be identified as

$$\rho_0 = \frac{1}{2}\gamma_{[ijk]}f_{[ijk]}. \tag{3.30}$$

The second term of (3.28) is the Freudenthal-de Vries' strange formula [7, 8],

$$\begin{aligned}(\rho, \rho)_0 &= \left(\frac{1}{2}\gamma_{[ijk]}f_{[ijk]}\right)^2 = \frac{1}{24}dim(g_0)g_0^\vee(\theta, \theta)_0 \\ &= \frac{1}{24}dim(g_0)C_2^{0,ad}.\end{aligned} \tag{3.31}$$

Where, in the above equation, $g_0^\vee$ is dual Coxeter number of g_0, θ is the highest root, and C_2^{ad} is the quadratic Casimir value in the adjoint representation.

Squaring (3.27), we get

$$\begin{aligned}(\not{\mathcal{K}}_\Lambda^1)^2 &= \frac{1}{2}\alpha_{[ab]}[F_a, F_b] + \frac{1}{2}\alpha_{(ab)}\{F_a, F_b\} \\ &\quad -\frac{1}{2}\gamma_i[\alpha_{(ab)}, \alpha_{a'}]f_{i[ab]}F_{a'} + \left(\frac{1}{2}\gamma_i\alpha_{(ab)}f_{i(ab)}\right)^2 \\ &= \frac{1}{2}\alpha_{[ab]}[F_a, F_b] + \frac{1}{2}\alpha_{(ab)}f_{i(ab)}L_i + \left(\frac{1}{2}\gamma_i\alpha_{(ab)}f_{i(ab)}\right)^2.\end{aligned} \tag{3.32}$$

In contrary to the even part, (3.32) by itself is not invariant due to the presence of a linear term in even generators. The linear term in L_i will be cancelled out by one of the the cross terms of the square of the combined even and odd Kostant's cubic Dirac operator,

$$\begin{aligned}
(\not{K}_\Lambda)^2 &= (\not{K}^0_\Lambda)^2 + (\not{K}^1_\Lambda)^2 + \{\gamma_i, \alpha_a\} L_i F_a - \frac{1}{2}\{\gamma_{[ijk]}, \alpha_a\} f_{[ijk]} F_a \\
&\quad -\frac{1}{2}\{\gamma_i, \gamma_{i'}\}\alpha_{(ab)} f_{i'(ab)} L_i - \frac{1}{2}\gamma_i[\alpha_{(ab)}, \alpha_{a'}] f_{i(ab)} F_{a'} \\
&\quad +\frac{1}{2}\{\gamma_{[ijk]}, \gamma_{i'}\}\alpha_{(ab)} f_{[ijk]} f_{i'(ab)} \\
&= \gamma_{(ij)} L_{(i} L_{j)} + \alpha_{[ab]} F_{[a} F_{b]} + \left(\frac{1}{2}\gamma_{[ijk]} f_{[ijk]}\right)^2 \\
&\quad + \left(\frac{1}{2}\gamma_i \alpha_{(ab)} f_{i(ab)}\right)^2 + \frac{1}{2}\gamma_{[ij]}\alpha_{(ab)} f_{[ijk]} f_{k(ab)} \qquad (3.33)
\end{aligned}$$

Since ρ_1, one-half the sum of positive odd roots, can be identified as

$$\rho_1 = -\frac{1}{2}\gamma_i \alpha_{(ab)} f_{i(ab)}. \qquad (3.34)$$

Recall that, for Lie superalgebras, $\rho = \rho_0 - \rho_1$. (3.33) can be simply written as

$$(\not{K}_\Lambda)^2 = C_2(\Lambda) + (\rho, \rho), \qquad (3.35)$$

where

$$C_2(\Lambda) = \gamma_{(ij)} L_{(i} L_{j)} + \alpha_{[ab]} F_{[a} F_{b]}.$$

The Laplacian operator turns out to be invariant under the action of even and odd generators of g again. For Lie superalgebras, the generalization of the Freudenthal-de Vries strange formula still holds [13]

$$(\rho, \rho) = \frac{g^\vee}{24}(\dim\, g_0 - \dim\, g_1), \tag{3.36}$$

where $g^\vee$'s, the dual Coxeter numbers, of g are given in Table 3.1.

g	$g^\vee$
$su(m\|n)$	$\|m-n\|$
$osp(2\|2n)$	n
$osp(2m+1\|n)$	$2(m-n)-1$ if $m>n$, $n-m+\frac{1}{2}$ if $m \leq n$
$osp(2m\|n)$	$2(m-n-1)$ if $m \geq n$, $n-m+1$ if $m<n+1$
$osp(4\|2;\alpha)$	0
$F(4)$	3
$G(3)$	2

Table 3.1. Dual Coxeter numbers of basic Lie superalgebras.

In an equal rank embedding of Lie superalgebras, $g \supset r$, with $\Phi_g \supset \Phi_r$, let the even and odd generators, L_i and F_a, span the basis of g. In according to the $g = r \oplus p$ decomposition, $L_{\hat{i}}$ and $F_{\hat{a}}$ span the basis of r and L_I and F_A span the basis of p, the orthogonal complement of r under the Killing form of g. The Kostant's cubic Dirac operator on $p = g/r$ is

$$\begin{aligned} \not{K}_p &= \not{K}_g - \not{K}_r \\ &= \gamma_I L_I + \alpha_A F_A - \frac{1}{2}\gamma_{[IJK]} f_{[IJK]} - \frac{1}{2}\gamma_I \alpha_{(AB)} f_{I(AB)}, \end{aligned} \tag{3.37}$$

where $f_{[IJK]}$ and $f_{I(AB)}$ are the structure constants of g that are not in r. Since, under the Killing form of g, r and p basis are orthogonal to each other,

$$\{\gamma_{\hat{i}}, \gamma_I\} = \{\gamma_{\hat{i}}, \alpha_A\} = \{\alpha_{\hat{a}}, \gamma_I\} = \{\alpha_{\hat{a}}, \alpha_A\} = 0.$$

As a result, we have

$$\{\not{\mathcal{K}}_r, \not{\mathcal{K}}_p\} = 0.$$

The square of Kostant's coset cubic Dirac operator simply is

$$\begin{aligned} (\not{\mathcal{K}}_p)^2 &= (\not{\mathcal{K}}_g)^2 - (\not{\mathcal{K}}_r)^2 \\ &= \left(C_2 + (\rho,\rho)\right)_g - \left(C_2 + (\rho,\rho)\right)_r . \end{aligned} \tag{3.38}$$

Notice that both $(\not{\mathcal{K}}_g)^2$ and $(\not{\mathcal{K}}_r)^2$ commute with the generators of r. Hence, $(\not{\mathcal{K}}_p)^2$ is also invariant under the action of r.

In conclusion, we have derived the Kac character formulas and have constructed Kostant's cubic Dirac operator for equal rank embeddings of Lie superalgebras. In case of reductive Lie algebras, the coset space method of equal rank embeddings led to a Kazama-Suzuki model construction of a new class of unitary $N = 2$ superconformal theories [5] and a subclass of the construction could be represented by Landau-Ginzburg models [14]. In case of Lie superalgebras, it deserves to be pursued whether there will be any relevance of the coset superspace method to a construction of any physical model.

CHAPTER 4
KAZAMA-SUZUKI CONSTRUCTION OF SUPERCONFORMAL FIELD THEORIES

Although a Dirac operator with a cubic term appeared as the supercurrent in the studies of the superconformal field theories and the supersymmetric Wess-Zumino-Witten action many years ago [5, 14, 15]. The recent study of equal rank embedding of semisimple complex Lie algebras [1, 2] led Kostant to determine an Euler multiplet as the kernel of the cubic Dirac operator and its square in general [3]. It was observed that the energy-momentum tensor appeared to be the Kac-Moody version of the square of the Kostant operator and the Euler multiplets turned to be the intrinsic ground states in the superconformal field theories [6].

In this chapter, we observe those facts from the view point of representation theory of Lie algebras and Kac-Moody algebras. The Kazama-Suzuki construction of the superconformal generators, supercurrent and energy-momentum tensor, is explained in section 4.1 and 4.2. The $N = 2$ superconformal field theories according to Kazama-Suzuki construction are classified by the equal rank embeddings in section 4.3. The Euler multiplet as the intrinsic ground state of superconformal field theories is shown in section 4.4.

4.1 Super-Kac-Moody algebras

By definition, Kac-Moody algebra is a class of infinite-dimensional Lie algebra with a circle as a central extension smoothly mapped to a finite-dimensional

Lie algebra. A super-Kac-Moody algebra $\hat{g}$ associated to the Lie algebra g and Lie group G is a Kac-Moody algebra over superspace (z, θ). With the supersymmetrized current decomposition over ordinary space z

$$J_A(z, \theta) = j_A(z) + \theta J_A(z), \tag{4.1}$$

the fermions $j_A(z)$ in the adjoint representation of g and $J_A(z)$ are not independent and satisfy the OPE's

$$\begin{aligned} j_A(z) j_B(0) &\sim \frac{\delta_{AB}}{z}, & (4.2)\\ J_A(z) j_B(0) &\sim j_A(z) J_B(0) \sim \frac{i f_{[ABC]} j_C(0)}{z}, & (4.3)\\ J_A(z) J_B(0) &\sim \frac{i f_{[ABC]} J_C(0)}{z} + \frac{k \delta_{AB}}{z^2}. & (4.4) \end{aligned}$$

The level k in (4.2) is absorbed into the fermion bilinear. To make the fermion and boson current generators be independent, Kazama and Suzuki [5] introduced the level $k = \hat{k} + g^\vee$ current decomposition

$$J_A^k(z) = \hat{J}_A^{\hat{k}}(z) + S_A = \hat{J}_A^{\hat{k}}(z) - \frac{i}{2} f_{[ABC]} j_{[B} j_{C]}(z) \tag{4.5}$$

so that $\hat{J}_A^{\hat{k}}(z)$ and $j_A(z)$ are mutually independent and, respectively, generate the level $\hat{k}$ and $g^\vee$ ordinary Kac-Moody algebras $\hat{g}$. The level $g^\vee$ Kac-Moody algebra $\hat{g}$ that is constructed by a bilinear form of the fermions in the adjoint representation of g turns out to be isomorphic to the level 1 ordinary Kac-Moody algebra $\widehat{so}(\dim g)$. Therefore, the level k super-Kac-Moody algebra can be regarded as a direct sum of two mutually independent ordinary Kac-Moody algebras

$$\hat{g}_k = \hat{g}_{\hat{k}} \oplus \widehat{so}(\dim\ g)_1. \tag{4.6}$$

In terms of oscillators, (4.5) is

$$\sum_n (J_n)_A z^{-n-1} = \sum_n (\hat{J}_n)_A z^{-n-1} - \frac{i}{2} f_{[ABC]} \sum_n (b_{-n})_B (b_n)_C z^{-1}. \tag{4.7}$$

In the representation theory of the ordinary Kac-Moody algebra, (4.7) corresponds to an affine weight decomposition

$$(\Lambda + \rho_g; \hat{k} + g^\vee; -n) = (\Lambda; \hat{k}; -n) + (\rho_g; g^\vee; 0), \tag{4.8}$$

where n is called a grade. When $n = 0$, the affine weight $(\Lambda; \hat{k}; 0)$ is called the highest weight. From the decomposition (4.6) of super-Kac-Moody algebra, there exists the associated $N = 1$ superconformal model with the superconformal generators

$$G(z) = (2/k)^{1/2} \left(j_A(z) \hat{J}_A(z) - \frac{i}{2} f_{[ABC]} j_{[A} j_B j_{C]}(z) \right), \tag{4.9}$$

and

$$T(z) = \frac{1}{k} \left(\hat{J}_A(z) \hat{J}_A(z) - j_A(z) \partial_z j_A(z) \right). \tag{4.10}$$

On the rhs of the holomorphic energy-momentum tensor $T(z)$, the first term commutes with the second term. Thus, the central charge of this superconformal model is the sum of the central charges of $\hat{g}_{\hat{k}}$ and $\widehat{so}(\dim g)_1$

$$c = \frac{\hat{k}\dim \mathrm{G}}{\hat{k} + g^{\vee}} + \frac{1}{2}\dim \mathrm{G}. \tag{4.11}$$

The superconformal generators $T(z)$ and $G(z)$ satisfy the OPE's

$$\begin{aligned} T(z)T(0) &\sim \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial_z T(0)}{z}, & (4.12) \\ T(z)G(0) &\sim G(z)T(0) \sim \frac{3G(0)/2}{z^2} + \frac{\partial_z G(0)}{z}, & (4.13) \\ G(z)G(0) &\sim \frac{2c/3}{z^3} + \frac{2T(0)}{z}. & (4.14) \end{aligned}$$

The underlying supersymmetric Wess-Zumino-Witten Lagrangian in two dimensions that posseses the super-Kac-Moody algebra $\hat{g}_k$ and the $N = 1$ superconformal symmetry was constructed in Ref. [15]. The supersymmetric WZW Lagrangian was shown to be equal to the original WZW Lagrangian of $\hat{g}_{\hat{k}}$ plus the Lagrangian of free fermions in the adjoint representation of $\hat{g}_k$. By applying Witten's non-Abelian bosonization rules to the WZW Lagrangian, the supersymmetric WZW theory is equivalent to two free fermion theories.

4.2 Kazama-Suzuki construction of $N = 1$ superconformal models

Instead of using Wick's theorem to calculate the OPE's to get the superconformal generators on the coset space, one can simply follow Brink and Ramond's canonical method [6]. The construction of superconformal generators from the coset of a super-Kac-Moody algebra can be done in analogy to section II.

On the world-sheet, there exists a natural generalization of γ-matrices and generators of a Lie algebra

$$\gamma_A \quad \rightarrow \quad j_A(z), \tag{4.15}$$

$$J_A \quad \rightarrow \quad \hat{J}_A(z). \tag{4.16}$$

The fermions $j_A(z)$ satisfy the anticommutator

$$\{j_A(z), j_B(0)\} = 2\delta_{AB}, \tag{4.17}$$

and the bosonic current generators $\hat{J}_A(z)$ satisfy the level $\hat{k}$ Kac-Moody commutator

$$[\hat{J}_i(z), \hat{J}_j(0)] = if_{[ijk]}\hat{J}_k(0) + \hat{k}\delta_{ij}\partial_z\delta(z), \tag{4.18}$$

$$[\hat{J}_i(z), \hat{J}_a(0)] = if_{[iab]}\hat{J}_b(0), \tag{4.19}$$

$$[\hat{J}_a(z), \hat{J}_b(0)] = if_{[abc]}\hat{J}_c(0) + if_{[abj]}\hat{J}_j(0) + \hat{k}\delta_{ab}\partial_z\delta(z). \tag{4.20}$$

All indices are the same as in section 2.2.

The supercurrent of $\hat{h}$ that lies in $\hat{g}$ is

$$G_{\hat{h}}(z) = \left(\frac{2}{k}\right)^{1/2} \left(j_i(\hat{J}_i - \frac{i}{2}f_{[iab]}j_{[a}j_{b]}) - \frac{i}{2}f_{[ijk]}j_{[i}j_j j_{k]}\right)(z), \tag{4.21}$$

and that of $\hat{g}/\hat{h}$ is

$$G_{\hat{g}/\hat{h}}(z) = G_{\hat{g}}(z) - G_{\hat{h}}(z) = \left(\frac{2}{k}\right)^{1/2} \left(j_a\hat{J}_a - \frac{i}{2}f_{[abc]}j_{[a}j_b j_{c]}\right)(z). \tag{4.22}$$

For simplicity, we restrict to the case where the coset space is symmetric, i.e. $f_{[abc]} = 0$. By using the anitcommutator (4.17) and the commutators (4.18), (4.19), and (4.20), one can show that squaring (4.22) gives rise to the energy-momentum tensor

$$\begin{aligned} T_{\hat{g}/\hat{h}}(z) &= \frac{1}{k}\left(\hat{J}_a\hat{J}_a - if_{[iab]}\hat{J}_{[i}j_a j_{b]} - \hat{k}j_a\partial_z j_a\right)(z) \\ &= \frac{1}{k}\left(\hat{J}_A^2 - (\hat{J}_i + S_i)^2 - \hat{k}j_a\partial_z j_a + S_i^2\right)(z), \end{aligned} \tag{4.23}$$

where $S_i(z) = -\frac{i}{2}f_{[iab]}j_{[a}j_{b]}(z)$. In comparison to (2.12), the energy-momentum tensor $T_{\hat{g}/\hat{h}}(z)$ is a generalization of Kostant operator $\not{\mathcal{K}}^2_{g/h}$. One can also see that OPE's

$$T_{\hat{g}/\hat{h}}(z)\hat{J}_i(0) \sim T_{\hat{g}/\hat{h}}(z)j_i(0) \sim 0 \tag{4.24}$$

imply the orthogonal decomposition of the superconformal generators

$$T_{\hat{g}/\hat{h}}(z)G_{\hat{h}}(0) \sim T_{\hat{g}/\hat{h}}(z)T_{\hat{h}}(0) \sim 0. \tag{4.25}$$

From (4.23), one can regard Kazama-Suzuki construction of equal rank embedding of super-Kac-Moody algebra as the extension of the GKO construction of ordinary Kac-Moody algebra

$$\left(\hat{g}_k/\hat{h}_k\right) \sim \left(\hat{g}_{\hat{k}}/\hat{h}_{\hat{k}+g^\vee-h^\vee}\right) \oplus \widehat{so}(\mathrm{dim}\ g/h)_1 \tag{4.26}$$

with central charge

$$\begin{aligned} c_{\hat{g}/\hat{h}} &= \frac{\hat{k}}{\hat{k}+g^\vee}\text{dim } g + \frac{1}{2}\text{dim } g/h - \frac{\hat{k}+g^\vee-h^\vee}{\hat{k}+g^\vee}\text{dim } h \\ &= \frac{3}{2}(\text{dim } g/h) - \frac{12}{(\hat{k}+g^\vee)}\left(|\rho_g|^2 - |\rho_h|^2\right). \end{aligned} \tag{4.27}$$

The equal rank embeddings in chapter 2 that can be used to construct the $N = 1$ superconformal models are:

- $F_4/SO(9)$ with central charge $c_{\hat{g}/\hat{h}} = 24\hat{k}/(\hat{k}+9)$.
- $Sp(2n+2)/Sp(2n) \times Sp(2)$ with central charge $c_{\hat{g}/\hat{h}} = 6\hat{k}n/(\hat{k}+n+2)$.

4.3 Kazama-Suzuki construction of $N = 2$ superconformal models

In the previous section, a class of $N = 1$ superconformal theories is obtained by an equal rank embedding of super-Kac-Moody algebras. In the case that the subalgebra has $u(1)$ factor, there exists a class of $N = 2$ superconformal theories. The equal rank embeddings in chapter 2 that are extented to construct the $N = 2$ superconformal models are listed in Table 4.1.

$\hat{g}/\hat{h}$	$c_{\hat{g}/\hat{h}}$
$\widehat{su}(n+1)/\widehat{su}(n) \times \hat{u}(1)$	$3\hat{k}n/(\hat{k}+n+1)$
$\widehat{so}(n+2)/\widehat{so}(n) \times \widehat{so}(2)$	$3\hat{k}n/(\hat{k}+n)$, $n \geq 2$
$\widehat{so}(3)/\widehat{so}(2) \sim \widehat{su}(2)/\hat{u}(1)$	$3\hat{k}/(\hat{k}+2)$
$\widehat{su}(m+n)/\widehat{su}(m) \times \widehat{su}(n) \times \hat{u}(1)$	$3\hat{k}mn/(\hat{k}+m+n)$
$\hat{E}_6/\widehat{so}(10) \times \hat{u}(1)$	$48\hat{k}/(\hat{k}+12)$

Table 4.1. Equal rank embeddings and their central charges $c_{\hat{g}/\hat{h}}$.

The $N = 2$ superconformal theories have a $u(1)$ current $U(z)$, two supercurrent $G_\pm(z)$ with $u(1)$ charges ± 1, and energy-momentum tensor $T(z)$ as their generators. These generators satisfy the OPE's

$$\begin{aligned}
T(z)T(0) &\sim \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial_z T(0)}{z}, \qquad (4.28)\\
T(z)U(0) &\sim \frac{U(0)}{z^2} + \frac{\partial_z U(0)}{z}, \qquad (4.29)\\
T(z)G_\pm(0) &\sim \frac{3G_\pm(0)/2}{z^2} + \frac{\partial_z G_\pm(0)}{z}, \qquad (4.30)\\
U(z)U(0) &\sim \frac{c/3}{z^2}, \qquad (4.31)\\
U(z)G_\pm(0) &\sim \frac{\pm G_\pm(0)}{z}, \qquad (4.32)\\
G_+(z)G_-(0) &\sim \frac{2c/3}{z^3} + \frac{2U(0)}{z^2} + \frac{2T(0) + \partial_z U(0)}{z}, \qquad (4.33)\\
G_+(z)G_+(0) &\sim G_-(z)G_-(0) \sim 0. \qquad (4.34)
\end{aligned}$$

The simplest representation of the $N = 2$ superconformal theories is the $\widehat{su}(2)/\hat{u}(1)$ minimal model with superconformal generators as follows:

$$\begin{aligned}
G_1(z) &= \left(\frac{2}{k}\right)^{1/2} \left(j_1 \hat{J}_1 + j_2 \hat{J}_2\right)(z), \qquad (4.35)\\
G_2(z) &= \left(\frac{2}{k}\right)^{1/2} \left(j_2 \hat{J}_1 - j_1 \hat{J}_2\right)(z), \qquad (4.36)\\
U(z) &= \frac{2}{k}\left(\hat{J}_3 + \frac{i\hat{k}}{k}\epsilon_{123} j_1 j_2\right)(z), \qquad (4.37)\\
T(z) &= \frac{1}{k}\left((\hat{J}_1)^2 + (\hat{J}_2)^2 - i j_1 j_2 \hat{J}_3 - \hat{k}(j_1 \partial_z j_1 + j_2 \partial_z j_2)\right)(z). \qquad (4.38)
\end{aligned}$$

The superconformal generators (4.35) and (4.38) are the generalization of the Kostant operator and its square of the $su(2)/u(1)$ model that appeared in section 2.2. At the end of the next chapter, we will derive these superconformal generators from the gauged supersymmetric Wess-Zumino-Witten Lagrangian density.

4.4 Ground states of superconformal theories

As discussed in section 4.1 and 4.2, the decomposition of $\hat{g}$ and $\hat{h}$ at level k are, respectively,

$$\begin{aligned} \hat{g}_k &= \hat{g}_{\hat{k}} \oplus \widehat{so}(\dim\, g)_1, \\ \hat{h}_k &= \hat{h}_{\hat{k}+g^\vee-h^\vee} \oplus \widehat{so}(\dim\, h)_1, \end{aligned} \tag{4.39}$$

and in terms of generators are, respectively,

$$\begin{aligned} J_A^k(z) &= \hat{J}_A^{\hat{k}}(z) \underbrace{-\frac{i}{2} f_{[ABC]} j_{[B} j_{C]}(z)}_{g^\vee}, \\ J_i^k(z) &= \hat{J}_i^{\hat{k}}(z) \underbrace{-\frac{i}{2} f_{[iab]} j_{[a} j_{b]}(z)}_{g^\vee - h^\vee} \underbrace{-\frac{i}{2} f_{[ijk]} j_{[j} j_{k]}(z)}_{h^\vee}. \end{aligned} \tag{4.40}$$

From the viewpoint of representation theory, the affine weights of $\hat{g}_k$ and $\hat{h}_k$ corresponding to (4.39) and (4.40) are, respectively,

$$\hat{\Lambda} + \hat{\rho}_g = (\Lambda; \hat{k}; -m) + (\rho_g; g^\vee; 0), \tag{4.41}$$

$$\hat{\lambda} + \hat{\rho}_h = (\lambda; \hat{k} + g^\vee - h^\vee; -n) + (\rho_h; h^\vee; 0). \tag{4.42}$$

Recall that an affine Weyl group of $\hat{g}$ is a semidirect product of the Weyl group of g by the group of translations by coroots. The affine Weyl group still preserves the affine scalar product [7, 8]. This allows us to define an Euler multiplet of $\hat{g}/\hat{h}$ as a set of the affine weights of $\hat{h}$ in accordance with the GKRS construction

$$\{c \bullet \hat{\Lambda} = c(\hat{\Lambda} + \hat{\rho}_g) - \hat{\rho}_h\}. \tag{4.43}$$

Now let us suppose that

$$\hat{\Lambda} + \hat{\rho}_g \geq \hat{w}^{-1}(\hat{\mu} + \hat{\rho}_h), \tag{4.44}$$

where $\hat{\mu}$ is an affine weight of $\hat{h}$ in the decomposition of $V_{\hat{\Lambda}} \otimes \hat{S}$ into irreducible representations of $\hat{h}$ and $\hat{w}$ is an element of the affine Weyl group such that $\hat{w}^{-1}(\hat{\mu}+\hat{\rho}_h)$ is a dominant weight. In analogy to the calculation of Lie algebras in section 2.3, one can show that (4.43) gives us, by using the identites in appendix C,

$$\frac{1}{2(\hat{k} + g^{\vee})}\left(C_2(\Lambda) - C_2(\mu) + |\rho_g|^2 - |\rho_h|^2\right) - m + n \geq 0. \tag{4.45}$$

As a result of (2.20), $m = n$. This means that in the Kazama-Suzuki construction if $\hat{\mu} = c \bullet \hat{\Lambda}$, the affine weight of $\hat{g}_{\hat{k}}$ at grade m always gives the affine weights of $\hat{h}_{\hat{k}+g^{\vee}-h^{\vee}}$ at the same grade m as the Euler multiplet. Adding and subtracting by the eigenvalue of the spinor highest weight of $\widehat{so}(\text{dim } g/h)_1$ to (4.45), we obtain

$$\underbrace{\frac{1}{2(\hat{k} + g^{\vee})}\left(C_2(\Lambda) - C_2(\mu)\right) + \frac{1}{2}\tilde{\Lambda}^2}_{\Delta\ (=\text{ conformal dimension})} \underbrace{- \frac{1}{16}(\text{dim } g/h) + \frac{1}{2(\hat{k} + g^{\vee})}\left(|\rho_g|^2 - |\rho_h|^2\right)}_{-c_{\hat{g}/\hat{h}}/24} \geq 0. \tag{4.46}$$

(4.46) corresponds to the vanishing of modular anomaly of the Kazama-Suzuki coset model and is the eigenvalue of zero mode of energy-momentum tensor (4.23). (4.46) also implies that the Euler multiplets are the intrinsic ground states of the superconformal field theories.

CHAPTER 5
THE GAUGED SUPERSYMMETRIC WZW LAGRANGIAN

In the previous chapter, the superconformal generators of the $SU(2)/U(1)$ superconformal model were derived from the superconformal algebra based on the Kazama-Suzuki construction. In this chapter, we will study a Lagrangian of $SU(2)/U(1)$ model as an underlying theory of the superconformal field theory.

Although the gauged supersymmetric Wess-Zumino-Witten action of the $N = 2$ $SU(2)/U(1)$ model was ready worked out in the Nakatsu's paper [16], a derivation of the superconformal generators, supercurrents and energy-momentum tensors, was not done. After reviewing the gauged supersymmetric Wess-Zumino-Witten action, we compute those superconformal generators by applying Noether's method to the $SU(2)/U(1)$ Lagrangian in the end of this chapter.

5.1 The gauged supersymmetric WZW action of a G/H model

The level $\hat{k}$ Wess-Zumino-Witten action at the infrared-stable fixed point is [17]

$$\begin{aligned} \frac{2\pi}{\hat{k}} S(U) &= \underbrace{\frac{1}{8}\int d^2x \eta^{\mu\nu}\,\mathrm{Tr}\{\partial_\mu U^{-1}\partial_\nu U\}}_{I_\sigma} \\ &\quad \underbrace{+\frac{1}{12}\int_B d^3y \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\{U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\}}_{I_{WZ}}, \end{aligned} \tag{5.1}$$

where the first integral is over a world-sheet with the Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1)$ and the second one is the volume integral bounded by the world-sheet.

In the light cone coordinates[1], $z = x_0 + x_1$, $\overline{z} = x_0 - x_1$, $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1})$, and $\partial_{\overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1})$, varying the WZW action yields two equivalent equations of motion

$$\partial_z \left(\partial_{\overline{z}} U U^{-1}\right) = \partial_{\overline{z}} \left(U^{-1} \partial_z U\right) = 0. \tag{5.2}$$

Now, consider a gauged supersymmetric WZW Lagrangian on a G/H model. Let $\{T^i,\ (T^{\overline{a}})^+,\ (T^{\overline{a}})^-\}$ be generators of $g = h \oplus t_+ \oplus t_-$, where $\{T^i\}$ generate h, $\{(T^{\overline{a}})^+\}$ generate t_+, and $\{(T^{\overline{a}})^-\}$ generate t_-. Replacing the partial derivative with covariant derivative,

$$\partial_\mu \to D_\mu = \partial_\mu - [A_\mu,\ \cdot\ ], \tag{5.3}$$

where A_μ are the H-valued gauge fields and adding the fermionic Lagrangian, one yields the gauged supersymmetric WZW action of a G/H model at level $\hat{k}$ is

$$\begin{aligned} \frac{2\pi}{\hat{k}} S(U, A, \psi) \;=\; & \frac{1}{8} \int d^2 x \eta^{\mu\nu}\ \mathrm{Tr}\{\partial_\mu U^{-1} \partial_\nu U\} \\ & + \frac{1}{12} \int d^3 y \epsilon^{\mu\nu\rho}\ \mathrm{Tr}\{g^{-1} \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\rho U\} \\ & + \frac{1}{4} \int d^2 x \eta^{\mu\nu}\ \mathrm{Tr}\{A_\mu \partial_\nu U U^{-1} - A_\nu U^{-1} \partial_\mu U + A_\mu U A_\nu U^{-1} - A_\mu A_\nu\} \\ & - i \int d^2 x\ \mathrm{Tr}\{\overline{\psi} \gamma^\mu (\partial_\mu \psi - [A_\mu, \psi])\}. \end{aligned} \tag{5.4}$$

---

[1] $\eta_{z\overline{z}} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\epsilon_{z\overline{z}} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The action (5.4) is in the supersymmetric form. It consists of the Wess-Zumino-Witten action belonging to the group G, the action of the fermions, and the interaction terms between the fermions and the currents associated to the group H. Varying (5.4), one obtains the equations of motion

$$D_\nu\left(U^{-1}D_\mu U\right) = F_{\nu\mu}, \tag{5.5}$$

$$D_\mu\left(D_\nu U U^{-1}\right) = F_{\mu\nu}, \tag{5.6}$$

and

$$\gamma^\mu D_\mu \psi = 0. \tag{5.7}$$

5.2 $N = 2$ superconformal symmetry of the G/H model

On the two-dimensional world-sheet, the γ-matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.8}$$

and the two-component spinor is

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}, \tag{5.9}$$

where $\psi^+ = \lambda^+T^+ + \lambda^-T^-$ and $\psi^- = \chi^+T^+ + \chi^-T^-$. [2]

---

[2] The sums over indices are suppressed.

The action (5.4) is invariant under the $N = 2$ superconformal field transformations. For holomorphic or left-moving part, those two sets of field transformations under infinitesimal Grassmannian parameters $\epsilon_\pm$ are

$$\begin{aligned}
\delta_+ U &= \epsilon_+(z)\lambda^+ U T^+, \\
\delta_+ \lambda^+ &= 0, \\
\delta_+ \lambda^- &= i\epsilon_+(z)\mathrm{Tr}\{(U^{-1}DU)T^-\}, \\
\delta_+ \chi^\pm &= 0,
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
\delta_- U &= \epsilon_-(z)\lambda^- U T^-, \\
\delta_- \lambda^- &= 0, \\
\delta_- \lambda^+ &= i\epsilon_-(z)\mathrm{Tr}\{(U^{-1}DU)T^+\}, \\
\delta_- \chi^\pm &= 0.
\end{aligned} \tag{5.11}$$

Similarly, for antiholomorphic or right-moving part, they are

$$\begin{aligned}
\overline{\delta}_+ U &= \overline{\epsilon}_+(\overline{z})\chi^+ T^+ U, \\
\overline{\delta}_+ \chi^+ &= 0, \\
\overline{\delta}_+ \chi^- &= i\overline{\epsilon}_+(\overline{z})\mathrm{Tr}\{(\overline{D}UU^{-1})T^-\}, \\
\overline{\delta}_+ \lambda^\pm &= 0,
\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}
\overline{\delta}_{-}U &= \overline{\epsilon}_{-}(\overline{z})\chi^{-}T^{-}U,\\
\overline{\delta}_{-}\chi^{-} &= 0,\\
\overline{\delta}_{-}\chi^{+} &= i\overline{\epsilon}_{-}(\overline{z})\mathrm{Tr}\{(\overline{D}UU^{-1})T^{+}\},\\
\overline{\delta}_{-}\lambda^{\pm} &= 0.
\end{aligned} \tag{5.13}$$

One can show that the fields $f = U$, λ, or χ preserve the $N = 2$ superconformal algebra

$$\begin{aligned}
\delta_{+}\delta'_{+}f &= \delta_{-}\delta'_{-}f = 0,\\
\overline{\delta}_{+}\overline{\delta}'_{+}f &= \overline{\delta}_{-}\overline{\delta}'_{-}f = 0,
\end{aligned} \tag{5.14}$$

and

$$[\delta_{+}, \delta_{-}]f = -i\epsilon_{+}\epsilon_{-}Df, \tag{5.15}$$

after using the gauge covariant equations of motion. (5.15) means that the commutator of two superconformal transformation is a conformal transformation. Similarly, the commutator of a conformal and superconformal transformation is a superconformal transformation. Thus, the conformal and superconformal transformations close to form the superconformal algebra.

5.3 The $SU(2)/U(1)$ model

Let $\sigma^i, i = 1, 2, 3$, be the Pauli matrices and let $T^0 = I$, $T^+ = \frac{1}{2}(\sigma^1 + i\sigma^2)$, $T^- = \frac{1}{2}(\sigma^1 - i\sigma^2)$, and $T^3 = \frac{1}{2}\sigma^3$ be the basis of fundamental fields, U, A, ψ, over $SU(2)$ manifold. In the $SU(2)/U(1)$ model,

$$U = U_0 I + iU_1\sigma^1 + iU_2\sigma^2 + iU_3\sigma^3 = \begin{pmatrix} U_0 + iU_3 & U_2 + iU_1 \\ -U_2 + iU_1 & U_0 - iU_3 \end{pmatrix}, \ \mathrm{det}U = 1,$$

$$A_z = A^3_z T^3,\ A_{\bar{z}} = A^3_{\bar{z}} T^3,$$

and

$$\psi^+ = \lambda^+ T^+ + \lambda^- T^-,\ \psi^- = \chi^+ T^+ + \chi^- T^-.$$

The gauged supersymmetric WZW action of the $SU(2)/U(1)$ is

$$\begin{aligned} \frac{2\pi}{\hat{k}} S(U, A, \chi^\pm, \lambda^\pm) \ = \ & \frac{1}{2}\int d^2x \ \mathrm{Tr}\{\partial_z U^{-1} \partial_{\bar{z}} U\} \\ & + \frac{1}{12}\int_B d^3y \epsilon^{\mu\nu\rho} \ \mathrm{Tr}\{U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\} \\ & + \int d^2x \ \mathrm{Tr}\{A^3_z T^3 \partial_{\bar{z}} U U^{-1} - A^3_{\bar{z}} T^3 U^{-1} \partial_z U \\ & + A^3_z A^3_{\bar{z}} T^3 U T^3 U^{-1} - A^3_z A^3_{\bar{z}} T^3 T^3\} \\ & - i\int d^x \{\chi^- \left(\partial_z - A^3_z\right) \chi^+ + \lambda^- \left(\partial_{\bar{z}} - A^3_{\bar{z}}\right) \lambda^+\}. \end{aligned} \tag{5.16}$$

For an infinitesimal parameter of the local $U(1)$ subgroup, $\epsilon = \epsilon^i(x)T^i = \epsilon^3(x)T^3$, the action $S(U, A, \chi, \lambda)$ is invariant under the local $U(1)$ gauge symmetry, [3]

$$\begin{aligned}
\delta U &= \epsilon^3(x)[T^3, U], \\
\delta A_z &= D_z\epsilon(x) = \partial_z\epsilon^3(x)T^3, \\
\delta A_{\bar{z}} &= D_{\bar{z}}\epsilon(x) = \partial_{\bar{z}}\epsilon^3(x)T^3, \\
\delta\chi^{\pm} &= \epsilon^3(x)[T^3, \chi^{\pm}], \\
\delta\lambda^{\pm} &= \epsilon^3(x)[T^3, \lambda^{\pm}].
\end{aligned}$$

The variation of U in terms of components is

$$\delta U_0 = \delta U_3 = 0,\ \delta U_1 = -\frac{i}{2}\epsilon^3(x)U_2,\ \delta U_2 = \frac{i}{2}\epsilon^3(x)U_1. \tag{5.17}$$

Because the gauge group is chosen to act freely, the gauge can be conveniently fixed by gauging away one of the off-diagonal components of U. This gauge choice is called a unitary gauge.

5.3.1 The $SU(2)/U(1)$ gauge-fixed Lagrangian

As previously derived in section 5.1, the equations of motion of $SU(2)/U(1)$ model one yields after varying (5.16) are

$$D_z\left(D_{\bar{z}}UU^{-1}\right) = \left(\partial_{\bar{z}}A^3_z - \partial_z A^3_{\bar{z}}\right)T^3 \tag{5.18}$$

$$D_{\bar{z}}\left(U^{-1}D_zU\right) = \left(\partial_{\bar{z}}A^3_z - \partial_z A^3_{\bar{z}}\right)T^3 \tag{5.19}$$

---

[3] $f \to f' = e^{\epsilon}fe^{-\epsilon} = f + [\epsilon, f] + \frac{1}{2!}[\epsilon, [\epsilon, f]] + \ldots$

$$\mathrm{Tr}\{\left(D_{\bar{z}}UU^{-1}\right)T^3\} + i\chi^-\chi^+ = 0 \tag{5.20}$$

$$\mathrm{Tr}\{\left(U^{-1}D_zU\right)T^3\} - i\lambda^-\lambda^+ = 0 \tag{5.21}$$

$$D_z\chi^\pm = D_{\bar{z}}\lambda^\pm = 0 \tag{5.22}$$

Let $U_2 = 0$ be the local $U(1)$ gauge fixing of U and let $U_0 = \phi_1$ and $U_3 = \phi_2$. So, we have

$$U_1 = (1 - \phi_1^2 - \phi_2^2)^{1/2} = (1 - |\phi|^2)^{1/2}, \tag{5.23}$$

and

$$U = \begin{pmatrix} \phi & i(1-|\phi|^2)^{1/2} \\ i(1-|\phi|^2)^{1/2} & \overline{\phi} \end{pmatrix}, \tag{5.24}$$

where $\phi = \phi_1 + i\phi_2$ and $\overline{\phi} = \phi_1 - i\phi_2$. For convenience in writing a gauge field, define the bosonic currents

$$J_B^+ = \phi_1\partial_z\phi_2 - \phi_2\partial_z\phi_1 = \frac{i}{2}(\phi\partial_z\overline{\phi} - \overline{\phi}\partial_z\phi), \tag{5.25}$$

$$J_B^- = -\left(\partial_{\bar{z}}\phi_1\phi_2 - \partial_{\bar{z}}\phi_2\phi_1\right) = -\frac{i}{2}\left(\partial_{\bar{z}}\phi\overline{\phi} - \partial_{\bar{z}}\overline{\phi}\phi\right), \tag{5.26}$$

and the fermionic currents

$$j_F^+ = \lambda^-\lambda^+, \tag{5.27}$$

$$j_F^- = \chi^-\chi^+. \tag{5.28}$$

Solving (5.20) and (5.21), one obtains the gauge fields

$$A_{\bar{z}}^3 = i\frac{J_B^- + j_F^-}{1 - |\phi|^2}, \tag{5.29}$$

and

$$A_z^3 = -i\frac{J_B^+ - j_F^+}{1-|\phi|^2}. \tag{5.30}$$

By substituting the fields, U and A, into (5.16), one yields the gauge-fixed Lagrangian density

$$\begin{aligned}\frac{2\pi}{\hat{k}}\mathcal{L}(\phi,\chi^{\pm},\lambda^{\pm}) &= \frac{1}{2}\frac{\left(\partial_z\overline{\phi}\partial_{\overline{z}}\phi + \partial_z\phi\partial_{\overline{z}}\overline{\phi}\right)}{1-|\phi|^2} - i\lambda^-\partial_{\overline{z}}\lambda^+ - i\chi^-\partial_z\chi^+ \\ &\quad - \frac{j_F^+J_B^- + j_F^-J_B^+ + j_F^+j_F^-}{1-|\phi|^2}.\end{aligned} \tag{5.31}$$

Note that in the unitary gauge the Wess-Zumino term is a total derivative and can be dropped out.

To obtain the explicit superconformal transformations which are the invariance of the gauge-fixed action, we write the Lagrangian density (5.31) as

$$\begin{aligned}\frac{2\pi}{\hat{k}}\mathcal{L}(\phi_{1,2},\chi_{1,2},\lambda_{1,2}) &= \frac{(\partial_z\phi_1\partial_{\overline{z}}\phi_1 + \partial_z\phi_2\partial_{\overline{z}}\phi_2)}{1-|\phi|^2} + i\sum_{i=1}^{2}(\lambda_i\partial_{\overline{z}}\lambda_i + \chi_i\partial_z\chi_i) \\ &\quad + \frac{2i(\partial_{\overline{z}}\phi_2\phi_1 - \partial_{\overline{z}}\phi_1\phi_2)}{1-|\phi|^2} + \frac{2i(\phi_1\partial_z\phi_2 - \phi_2\partial_z\phi_1)}{1-|\phi|^2} \\ &\quad + \frac{4\lambda_1\lambda_2\chi_1\chi_2}{1-|\phi|^2}.\end{aligned} \tag{5.32}$$

In case $\phi_1 = \sin r\cos t$ and $\phi_2 = \sin r\sin t$, the first term on the rhs of (5.32) is the two dimensional black hole metric

$$\frac{(\partial_z\phi_1\partial_{\overline{z}}\phi_1 + \partial_z\phi_2\partial_{\overline{z}}\phi_2)}{1-|\phi|^2} = \partial_z r\partial_{\overline{z}}r + \tan^2 r\partial_z t\partial_{\overline{z}}t. \tag{5.33}$$

5.3.2 $N = 2$ superconformal transformations

In the holomorphic sector, the fields transform under the infinitesimal Grassmannian constants ϵ_1 as

$$\begin{aligned}
\delta_1\phi_1 &= \epsilon_1(1-|\phi|^2)^{1/2}\lambda_1, \\
\delta_1\phi_2 &= \epsilon_1(1-|\phi|^2)^{1/2}\lambda_2, \\
\delta_1\lambda_1 &= \epsilon_1(1-|\phi|^2)^{-1/2}(i\partial_z\phi_1-\phi_2\lambda_1\lambda_2), \\
\delta_1\lambda_2 &= \epsilon_1(1-|\phi|^2)^{-1/2}(i\partial_z\phi_2+\phi_1\lambda_1\lambda_2), \\
\delta_1\chi_1 &= \epsilon_1(1-|\phi|^2)^{-1/2}(\phi_1\lambda_2-\phi_2\lambda_1)\chi_2, \\
\delta_1\chi_2 &= -\epsilon_1(1-|\phi|^2)^{-1/2}(\phi_1\lambda_2-\phi_2\lambda_1)\chi_1,
\end{aligned} \tag{5.34}$$

and under the infinitesimal Grassmannian constants ϵ_2 as

$$\begin{aligned}
\delta_2\phi_1 &= \epsilon_2(1-|\phi|^2)^{1/2}\lambda_2, \\
\delta_2\phi_2 &= -\epsilon_2(1-|\phi|^2)^{1/2}\lambda_1, \\
\delta_2\lambda_1 &= -\epsilon_2(1-|\phi|^2)^{-1/2}(i\partial_z\phi_2+\phi_1\lambda_1\lambda_2), \\
\delta_2\lambda_2 &= \epsilon_2(1-|\phi|^2)^{-1/2}(i\partial_z\phi_1-\phi_2\lambda_1\lambda_2), \\
\delta_2\chi_1 &= -\epsilon_2(1-|\phi|^2)^{-1/2}(\phi_1\lambda_1+\phi_2\lambda_2)\chi_2, \\
\delta_2\chi_2 &= \epsilon_2(1-|\phi|^2)^{-1/2}(\phi_1\lambda_1+\phi_2\lambda_2)\chi_1.
\end{aligned} \tag{5.35}$$

In the antiholomorphic sector, the fields transform under the infinitesimal Grassmannian constants $\bar{\epsilon}_1$ as

$$\begin{aligned}
\bar{\delta}_1\phi_1 &= \bar{\epsilon}_1(1-|\phi|^2)^{1/2}\chi_1, \\
\bar{\delta}_1\phi_2 &= \bar{\epsilon}_1(1-|\phi|^2)^{1/2}\chi_2, \\
\bar{\delta}_1\chi_1 &= \bar{\epsilon}_1(1-|\phi|^2)^{-1/2}(i\partial_{\bar{z}}\phi_1-\phi_2\chi_1\chi_2), \\
\bar{\delta}_1\chi_2 &= \bar{\epsilon}_1(1-|\phi|^2)^{-1/2}(i\partial_{\bar{z}}\phi_2+\phi_1\chi_1\chi_2), \\
\bar{\delta}_1\lambda_1 &= \bar{\epsilon}_1(1-|\phi|^2)^{-1/2}(\phi_1\chi_2-\phi_2\chi_1)\lambda_2, \\
\bar{\delta}_1\lambda_2 &= -\bar{\epsilon}_1(1-|\phi|^2)^{-1/2}(\phi_1\chi_2-\phi_2\chi_1)\lambda_1,
\end{aligned} \tag{5.36}$$

and under the infinitesimal Grassmannian constants $\bar{\epsilon}_2$ as

$$\begin{aligned}
\bar{\delta}_2\phi_1 &= \bar{\epsilon}_2(1-|\phi|^2)^{1/2}\chi_2, \\
\bar{\delta}_2\phi_2 &= -\bar{\epsilon}_2(1-|\phi|^2)^{1/2}\chi_1, \\
\bar{\delta}_2\chi_1 &= -\bar{\epsilon}_2(1-|\phi|^2)^{-1/2}(i\partial_{\bar{z}}\phi_2+\phi_1\chi_1\chi_2), \\
\bar{\delta}_2\chi_2 &= \bar{\epsilon}_2(1-|\phi|^2)^{-1/2}(i\partial_{\bar{z}}\phi_1-\phi_2\chi_1\chi_2), \\
\bar{\delta}_2\lambda_1 &= -\bar{\epsilon}_2(1-|\phi|^2)^{-1/2}(\phi_1\chi_1+\phi_2\chi_2)\lambda_2, \\
\bar{\delta}_2\lambda_2 &= \bar{\epsilon}_2(1-|\phi|^2)^{-1/2}(\phi_1\chi_1+\phi_2\chi_2)\lambda_1.
\end{aligned} \tag{5.37}$$

One can directly calculate and verify that these field transformations satisfy the superconformal symmetry

$$[\delta_1, \delta_2]f = -2i\epsilon_1\epsilon_2\partial_{\bar{z}}f, \tag{5.38}$$

where $f = \phi_i, \lambda_i, \chi_i$.

5.4 The $SU(2)/U(1)$ superconformal generators

In general quantum field theory, when a field gets changed by a total derivative

$$\delta\varphi = a^\mu \partial_\mu \varphi, \tag{5.39}$$

$$\delta\partial_\mu\varphi = a^\nu \partial_\nu \partial_\mu \varphi, \tag{5.40}$$

the Lagrangian is also changed by a total derivative

$$\delta\mathcal{L} = a^\mu \partial_\mu \mathcal{L} = \frac{\delta\mathcal{L}}{\delta\varphi} a^\mu \partial_\mu \varphi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi} a^\nu \partial_\nu \partial_\mu \varphi. \tag{5.41}$$

By using the equation of motion

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} \right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0, \tag{5.42}$$

(5.41) becomes

$$a^\mu \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi} \partial^\nu \varphi - \eta^{\mu\nu} \mathcal{L} \right) = 0 = a^\mu \partial_\mu T^{\mu\nu}, \tag{5.43}$$

where the conserved energy-momentum tensor $T^{\mu\nu}$ is defined as

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi} \partial^\nu \varphi - \eta^{\mu\nu} \mathcal{L}. \tag{5.44}$$

The energy-momentum tensor in the holomorphic sector is

$$T(z) = 2\pi T_{zz} = \pi \frac{T^{\bar{z}\bar{z}}}{2} = \frac{\partial \mathcal{L}}{\partial \partial_{\bar{z}} \varphi} \partial_z \varphi, \tag{5.45}$$

and in the antiholomorphic sector is

$$\overline{T}(\bar{z}) = 2\pi T_{\bar{z}\bar{z}} = \pi \frac{T^{zz}}{2} = \frac{\partial \mathcal{L}}{\partial \partial_z \varphi} \partial_{\bar{z}} \varphi. \tag{5.46}$$

When the field gets changed proportional to a small parameter ϵ, a variation of the Lagrangian density gives the conserved current, after using the equation of motion,

$$\partial_\mu J^\mu = 0, \tag{5.47}$$

where

$$J^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \frac{\delta \varphi}{\epsilon}. \tag{5.48}$$

In the case that the action is supersymmetric, we obtain the supercurrents in the holomorphic sector as

$$G(z) = 2\pi \frac{\partial \mathcal{L}}{\partial \partial_{\bar{z}} \varphi} \frac{\delta \varphi}{\epsilon} \tag{5.49}$$

and in the antiholomorphic sector as

$$\overline{G}(\bar{z}) = 2\pi \frac{\partial \mathcal{L}}{\partial \partial_z \varphi} \frac{\overline{\delta} \varphi}{\bar{\epsilon}}. \tag{5.50}$$

Now, we are ready to compute the supercurrents and energy-momentum tensors of the $SU(2)/U(1)$ model from the Lagrangian density (5.32).

- Holomorphic sector:

$$\begin{aligned}
\frac{1}{\hat{k}}G_1(z) &= \frac{1}{\left(1-|\phi|^2\right)^{1/2}}\left(\lambda_1\partial_z\phi_1+\lambda_2\partial_z\phi_2\right), && (5.51)\\
\frac{1}{\hat{k}}G_2(z) &= \frac{1}{\left(1-|\phi|^2\right)^{1/2}}\left(\lambda_2\partial_z\phi_1-\lambda_1\partial_z\phi_2\right), && (5.52)\\
\frac{2}{\hat{k}}T(z) &= \frac{(\partial_z\phi_1)^2}{1-|\phi|^2}+\frac{(\partial_z\phi_2)^2}{1-|\phi|^2}-2i\lambda_1\lambda_2\frac{(\phi_1\partial_z\phi_2-\phi_2\partial_z\phi_1)}{1-|\phi|^2}\\
&\quad +i\left(\lambda_1\partial_z\lambda_1+\lambda_2\partial_z\lambda_2\right). && (5.53)
\end{aligned}$$

- Antiholomorphic sector:

$$\begin{aligned}
\frac{1}{\hat{k}}\overline{G}_1(\overline{z}) &= \frac{1}{\left(1-|\phi|^2\right)^{1/2}}\left(\chi_1\partial_{\overline{z}}\phi_1+\chi_2\partial_{\overline{z}}\phi_2\right), && (5.54)\\
\frac{1}{\hat{k}}\overline{G}_2(\overline{z}) &= \frac{1}{\left(1-|\phi|^2\right)^{1/2}}\left(\chi_2\partial_{\overline{z}}\phi_1-\chi_1\partial_{\overline{z}}\phi_2\right), && (5.55)\\
\frac{2}{\hat{k}}\overline{T}(\overline{z}) &= \frac{(\partial_{\overline{z}}\phi_1)^2}{1-|\phi|^2}+\frac{(\partial_{\overline{z}}\phi_2)^2}{1-|\phi|^2}-2i\chi_1\chi_2\frac{(\phi_1\partial_{\overline{z}}\phi_2-\phi_2\partial_{\overline{z}}\phi_1)}{1-|\phi|^2}\\
&\quad +i\left(\chi_1\partial_{\overline{z}}\chi_1+\chi_2\partial_{\overline{z}}\chi_2\right). && (5.56)
\end{aligned}$$

Up to some multiplicative factors, these superconformal generators are similar to those shown at the end of section 4.3.

CHAPTER 6
CONCLUSION

We studied the GKRS construction of an Euler multiplet of semisimple complex Lie algebras and extended the construction to basic Lie superalgebras. We observed that the Kostant's cubic Dirac operators of these two algebras are in a similar form and, in the case of semisimple Lie algebras, are analog to the supercurrents in superconformal field theories.

At the level of representation theory, we studied the analogy between the GKRS construction of the Euler multiplet of semisimple complex Lie algebras and the Kazama-Suzuki construction of super-Kac-Moody algebras associated with superconformal field theories. The Euler multiplets constructed by the GKRS method were found to be the intrinsic ground states of superconformal theories. The minimum eigenvalues of the Kostant operator in the GKRS construction and of energy-momentum tensor in the Kazama-Suzuki construction were derived from the Euler multiplet and were shown to be zero.

We also studied the gauged supersymmetric Wess-Zumino-Witten Lagrangian and derived the superconformal generators, supercurrents and energy-momentum tensors, in the $SU(2)/U(1)$ case.

APPENDIX A
DIMENSIONAL FORMULAS OF $SO(9)$ AND F_4

To calculate the dimensions of the irreducible representations of $SO(9)$ and F_4 in Table 2.1, use the Weyl dimensional formula [8, 7]. We obtain as follows:

- $SO(9)$ dimensional formula

$$\begin{aligned} I_0(\lambda) &= (1+\tilde{a}_1)(1+\tilde{a}_2)(1+\tilde{a}_3)(1+\tilde{a}_4) \\ &\quad (2+\tilde{a}_1+\tilde{a}_2)(2+\tilde{a}_2+\tilde{a}_3)(2+\tilde{a}_3+\tilde{a}_4)/8 \\ &\quad (3+2\tilde{a}_3+\tilde{a}_4)(3+\tilde{a}_1+\tilde{a}_2+\tilde{a}_3)(3+\tilde{a}_2+\tilde{a}_3+\tilde{a}_4)/27 \\ &\quad (4+\tilde{a}_1+\tilde{a}_2+\tilde{a}_3+\tilde{a}_4)(4+\tilde{a}_2+2\tilde{a}_3+\tilde{a}_4)/16 \\ &\quad (5+2(\tilde{a}_2+\tilde{a}_3)+\tilde{a}_4)(5+\tilde{a}_1+\tilde{a}_2+2\tilde{a}_3+\tilde{a}_4)/25 \\ &\quad (7+2(\tilde{a}_1+\tilde{a}_2+\tilde{a}_3)+\tilde{a}_4)(6+\tilde{a}_1+2(\tilde{a}_2+\tilde{a}_3)+\tilde{a}_4)/42. \end{aligned} \tag{A.1}$$

- F_4 dimensional formula

$$\begin{aligned} I_0(\Lambda) &= (1+a_1)(1+a_2)(1+a_3)(1+a_4) \\ &\quad (2+a_4+a_3)(2+a_3+a_2)(2+a_2+a_1)/8 \\ &\quad (3+a_4+a_3+a_2)(3+a_3+2a_2)(3+a_3+a_2+a_1)/27 \\ &\quad (4+a_4+a_3+a_2+a_1)(4+a_3+2a_2+a_1) \\ &\quad (4+a_4+a_3+2a_2)/64 \end{aligned}$$

$$
\begin{aligned}
&(5 + a_4 + a_3 + 2a_2 + a_1)(5 + a_4 + 2(a_3 + a_2)) \\
&(5 + a_3 + 2(a_2 + a_1))/125 \\
&(6 + a_4 + 2(a_3 + a_2) + a_1)(6 + a_4 + a_3 + 2(a_2 + a_1))/36 \\
&(7 + a_4 + 2(a_3 + a_2 + a_1))(7 + a_4 + 2a_3 + 3a_2 + a_1)/49 \\
&(8 + a_4 + 2a_3 + 3a_2 + 2a_1)/8 \\
&(9 + a_4 + 2a_3 + 4a_2 + 2a_1)/9 \\
&(10 + a_4 + 3a_3 + 4a_2 + 2a_1)/10 \\
&(11 + 2a_4 + 3a_3 + 4a_2 + 2a_1)/11. \qquad \text{(A.2)}
\end{aligned}
$$

The numerical values of dimensions of $SO(9)$ and F_4 irreducible representations were also provided by Ref. [10].

APPENDIX B
BRANCHING ALGORITHM

A routine for branching a Lie algebra g to its subalgebra h can be written by using the built-in functions in the LiE, a software package for Lie group theoretical computations developed by Arjeh M. Cohen, Marc van Leeuwen and Bert Lisser. The LiE package and its manual are freely distributed and can be obtained at http://wallis.univ-poitiers.fr/~maavl.

The fundamental concept of a branching algorithm is as follows:

1. Generating all the weights of an irrep of g.
2. Projecting all g-weights to h-weights by the projection matrix.
3. Selecting only the dominant weights of h and assembling them into the irreps of h.

The branching from $SO(32)$ to $SO(10) \times SO(2)$ is shown below as an example in the LiE programming language.

```
###########################################
#A routine for branching an irrep of SO(32) to the irreps of SO(10) × SO(2).
#Input parameters: dynkin=X[...]
###########################################
print(”\newline”)
print(dynkin)
print(dim(dynkin,D16))
print("\newline")
so32_dom = dom_char(dynkin,D16)
proj =[[0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0],
        [0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0],
        [0, 1, 0, 0, 0, 0, 1, 2, 1, 0, 0, 0, 0, 1, 0, 0],
        [1, 0, 0, 0, 0, 0, 0, 0, 1, 2, 1, 2, 1, 0, 0, 0],
        [0, 0, 1, 2, 1, 2, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0],
        [1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14, 7, 8]]
poly_so10so2_dom = X([0,0,0,0,0,0])
for i = 1 to length(so32_dom) do
    so32_orb_size = W_orbit_size(expon(so32_dom, i),D16);
    so10so2_states = null(so32_orb_size, 6);
    so32_states = W_orbit(expon(so32_dom, i),D16);
    for j = 1 to so32_orb_size do
        so10so2_states[j] = proj * so32_states[j];
        if (j == so32_orb_size) then
        so10so2_dom = filter_dom(so10so2_states,D5T1);
        poly_so10so2_dom = poly_so10so2_dom
                                + (coef(so32_dom, i) * X(so10so2_dom));
        fi;
    od;
od
```

```
so10so2_irrps = v_decomp(poly_so10so2_dom,D5T1)
for i = 1 to length(so10so2_irrps) do
    print(coef(so10so2_irrps, i)*X(expon(so10so2_irrps, i)));
    print(dim(expon(so10so2_irrps, i),D5T1));
od
print("\newline")
############################################
#End of routine.
############################################
```

APPENDIX C
USEFUL IDENTITES OF AFFINE LIE ALGEBRAS

Let affine weight $\hat{\lambda} = (\lambda; k; -m)$ and affine Weyl vector $\hat{\rho} = (\rho; g^\vee; 0)$.

- The inner product of the affine weights: $(\hat{\lambda}, \hat{\lambda}') = (\lambda, \lambda') - km' - k'm$.
- $|\hat{\rho}|^2 = (\hat{\rho}, \hat{\rho}) = (\rho, \rho) = \frac{g^\vee}{12}\text{dim } g$.
- $|\hat{\lambda}+\hat{\rho}|^2 = (\hat{\lambda}+\hat{\rho}, \hat{\lambda}+\hat{\rho}) = (\hat{\lambda}, \hat{\lambda}+2\hat{\rho})+(\hat{\rho}, \hat{\rho}) = (\lambda, \lambda+2\rho)-2m(k+g^\vee)+(\rho, \rho)$.
- $C_2(\hat{\lambda}) = (\hat{\lambda}, \hat{\lambda}+2\hat{\rho}) = (\lambda, \lambda+2\rho) - 2m(k+g^\vee) = \frac{I_2}{I_0}\text{dim } g - 2m(k+g^\vee)$.
- Conformal dimension: $\Delta_{\hat{\lambda}} = \frac{(\lambda,\lambda+2\rho)}{2(k+g^\vee)} - m$.
- Modular anomaly of $\hat{\lambda}$:

$$m_{\hat{\lambda}} = \frac{|\hat{\lambda}+\hat{\rho}|^2}{2(k+g^\vee)} - \frac{|\hat{\rho}|^2}{2g^\vee} = \frac{|\lambda+\rho|^2}{2(k+g^\vee)} - \frac{|\rho|^2}{2g^\vee} - m = \Delta_{\hat{\lambda}} - \frac{c}{24} - m.$$

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BIOGRAPHICAL SKETCH

Teparksorn Pengpan was born at Sakon Nakorn, Thailand, on December 15, 1964, the second son. His parents had three sons and two daughters. In the deepest side of his mind, he always wishes to see everyone in his family alive and happy to admire his achievement in studying abroad. Sadly, because of hard work and cancer, his brother died on June 24, 1999. This dissertation is dedicated to be one of the memorials to his brother.

After completing a M.Sc. degree in chemical physics from Mahidol University in 1987, he began to work as a lecturer at the Prince of Songkla University. Since, his country is still developing and there is a lack of human resources in the natural sciences, one solution is to develop a proficient teaching faculty in the universities. On the one hand, he was lucky to be granted a Thai government scholarship to study abroad. He first entered a Ph.D. program in physics at Kent State University in fall 1995 and then transferred the program to the University of Florida in fall 1996. Here, his longing to understand about Lie algebras and quantum field theories was fulfilled at some level. On the other hand, his Ph.D. study was quite painful. However, it is an invaluable experience that one should not forget and it brings happiness to life in the end.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

P. Ramond, Chairman
Distinguished Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Z. Qiu
Associate Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

John Klauder
J. Klauder
Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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This dissertation was submitted to the Graduate Faculty of the Department of Physics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirments for the degree of Doctor of Philosophy.

August 2000

Dean, Graduate School